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Nonlinear vibrations of axially moving Timoshenko beams under weak and strong external excitations

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Abstract

In this paper, nonlinear vibrations under weak and strong external excitations of axially moving beams are analyzed based on the Timoshenko model. The governing nonlinear partial-differential equation of motion is derived from Newton's second law, accounting for the geometric nonlinearity caused by finite stretching of the beams. The complex mode approach is applied to obtain the transverse vibration modes and the natural frequencies of the linear equation. The method of multiple scales is employed to investigate primary resonances, nonsyntonic excitations, superharmonic resonances, and subharmonic resonances. Some numerical examples are presented to demonstrate the effects of a varying parameter, such as axial speed, external excitation amplitudes, and nonlinearity, on the response amplitudes for the first and second modes, when other parameters are fixed. The stability of the response amplitudes is investigated and the boundary of instability is located.

1. Introduction

Many real-life engineering devices, such as band saws, power transmission chains, aerial cableways, and serpentine belts, involve the transverse vibration of axially moving beams. Despite its wide applications, these devices suffer from the occurrence of large transverse vibrations due to initial excitations. Transverse vibrations of these devices have been investigated to avoid possible resulting fatigue, failure, and low quality. For example, the vibration of the blade of band saws causes poor cutting quality. The vibration of the belt leads to noise and accelerated wear of the belt in belt drive systems. Therefore, vibration analysis of axially moving beams is important for the design of the devices.

There are many researches that have been carried out on axially moving systems in literatures. Mote [1] first investigated the effect of tension in an axially moving band and computed numerically the first three frequencies and modes for simply supported boundary conditions. Wu and Mote [2] studied the linear

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problem of parametric excitation of an axially moving band by periodic edge loading with different assumptions. Wickert and Mote [3] presented a classical vibration theory, comprised of a modal analysis and a Green's function method, for the traveling string and the traveling beam, where natural frequencies and modes associated with free vibration serve as a basis for analysis. Öz and Pakdemirli [4] and Öz [5] calculated the first two natural frequency values for different flexural stiffnesses in the cases of pinned–pinned ends and clamped–clamped ends, respectively. Ghayesh and Khadem [6] investigated free nonlinear transverse vibration of an axially moving beam in which rotary inertia and temperature variation effects have been considered and they gave natural frequency versus the mean velocity and rotary inertia, critical speed versus the rotary inertia, and natural frequency versus the mean velocity and temperature for the first two modes. Lee and Jang [7] investigated the effects of continuously incoming and outgoing semi-infinite beam parts on dynamic characteristics and stability of an axially moving beam by using the spectral element method. In all of the above literatures, either simple support or fixed support is researched. Yang and Chen [8] studied axially moving elastic beams, and Chen and Yang [9] studied viscoelastic beams, on simple supports with torsion springs and gave the first two frequencies and modes.

Lee et al. [10] used exact dynamic-stiffness matrix in structural dynamics to provide very accurate solutions, while reducing the number of degrees of freedom to resolve the computational and cost problems. Chakraborty and Mallik [11] used wave propagation in a simply supported traveling beam to derive forced responses. Zhang and Zu [12] investigated the nonlinear forced vibration of viscoelastic moving belts excited by the eccentricity of pulleys. Pellicano and Vestroni [13] investigated the dynamic response of a simply supported traveling beam subjected to a transverse load in the super-critical speed range. Yang and Chen [14] studied the nonlinear forced vibration of axially moving viscoelastic beams excited by vibration of the supporting foundation.

The axially moving beam has been traditionally represented by the Euler–Bernoulli beam theory by assuming that the beam is relatively thin compared to its length. It appears that, to the authors' knowledge, there have been very few studies on the axially moving beam for the Timoshenko model. Simpson [15] was probably the first to derive equations of motion for the moving thick beam on the basis of the Timoshenko beam theory, but no numerical results were given and he did not consider axial tension in his equations. Chonan [16] studied the steady-state response of a moving Timoshenko beam by applying the Laplace transform method. Arboleda-Monsalve et al. [17] presented the dynamic-stiffness matrix and load vector of a Timoshenko beam column resting on a two-parameter elastic foundation with generalized end conditions. Mei et al. [18] presented wave vibration analysis of an axially loaded cracked Timoshenko beam.

The present paper is organized as follows. Section 2 derives the governing nonlinear partial-differential equation of motion. Section 3 employs the method of multiple scales to investigate nonlinear vibrations under weak and strong external excitations. Section 4 ends the paper with concluding remarks.

2. The governing equation

Uniform axially moving Timoshenko beams, with density ρ, cross-sectional area A, area moment of inertia of the cross-section about the neutral axis J, initial tension P, shape factor k, modulus of elasticity E, axial tension N, beam shearing modulus G, travel at the constant axial transport speed Γ between two simple supports separated by distance L under the distributed external excitation F in the transverse direction. The bending vibration can be described by two variables dependent on axial coordinate X and time T, namely, transverse displacement $Y(X,T)$ and angle of rotation of the beam cross-section $\Psi(X,T)$ due to the bending moment.

Because shear deformations are considered, the angle of the beam θ depends not only on the angle Ψ but also on the shear angle γ:

$$\theta(x,t) = \psi(x,t) - \gamma(x,t) \tag{1}$$

The bending moment $M(x,t)$ and shear force $Q(x,t)$ are related to the corresponding deformations:

$$M = EJ\psi_{,x},$$
$$Q = M_{,x} = \frac{AG}{k}(\psi - \theta) = \frac{AG}{k}(\psi - Y_{,x}) \tag{2}$$

Coupled governing equations are obtained according to Newton's second law:

$$\begin{aligned}&-Q_{,X}\,\mathrm{d}X\cos\theta-\rho A(Y_{,TT}+2\Gamma Y_{,XT}\cos\theta+\Gamma^2 Y_{,XX}\cos\theta)\,\mathrm{d}X\\&\quad+N\sin\theta+N_{,X}\,\mathrm{d}X\sin\theta+F\,\mathrm{d}X=0,\\&\rho J\psi_{,TT}-M_{,X}+Q=0\end{aligned}\tag{3}$$

where F is the external transverse excitation. The axial tension N is composed of the initial tension and the tension due to the transverse displacement:

$$N=P+\tfrac{1}{2}EA(Y_{,X})^2\tag{4}$$

Substituting Eqs. (2) and (4) into Eq. (3) and using the approximate expressions $\cos\theta=1$ and $\sin\theta=y_{,x}$ yield the governing equation for the transverse vibration of axially moving Timoshenko beams:

$$\begin{aligned}&\rho J\left(1+\frac{kP}{AG}+\frac{kE}{G}\right)Y_{,XXTT}-EJ\left(1+\frac{kP}{AG}\right)Y_{,XXXX}+PY_{,XX}+\frac{3}{2}EAY_{,X}^2Y_{,XX}\\&-\frac{\rho^2 Jk}{G}(Y_{,TTTT}+\Gamma^2Y_{,XXTT}+2\Gamma Y_{,XTTT})+\frac{3\rho EJk}{2G}(Y_{,X}^2Y_{,XXTT}+4Y_{,X}Y_{,XT}Y_{,XXT}\\&+2Y_{,X}Y_{,XX}Y_{,XTT}+2Y_{,XT}^2Y_{,XX})+\frac{\rho EJk}{G}(2\Gamma Y_{,XXXT}+\Gamma^2Y_{,XXXX})-\frac{3E^2Jk}{2G}(Y_{,X}^2Y_{,XXXX}\\&+6Y_{,X}Y_{,XX}Y_{,XXX}+2Y_{,XX}^3)-\rho A(Y_{,TT}+2\Gamma Y_{,XT}+\Gamma^2Y_{,XX})+F=0\end{aligned}\tag{5}$$

The boundary conditions for the simple supports at both ends are

$$Y|_{X=0}=0,\quad Y|_{X=L}=0;\quad EJ\psi_{,X}\big|_{X=0}=0,\quad EJ\psi_{,X}\big|_{X=L}=0\tag{6}$$

It is assumed that the external transverse excitation is a spatially uniformly distributed periodic force as $F=F_1\cos(\Omega T)$, where F_1 is the excitation amplitude and Ω the excitation frequency.

Introduce the dimensionless variables and parameters

$$\begin{aligned}&y=\frac{Y}{\varepsilon^{0.5}L},\quad x=\frac{X}{L},\quad t=T\sqrt{\frac{P}{\rho AL^2}},\quad v=\Gamma\sqrt{\frac{\rho A}{P}},\quad k_0=\frac{kJP}{GA^2L^2},\quad k_1=\frac{kJE}{GAL^2},\\&k_2=\frac{EA}{P},\quad k_3=\frac{J}{AL^2},\quad k_4=\frac{EJ}{PL^2},\quad \omega=\Omega\sqrt{\frac{\rho AL^2}{P}},\quad b=\frac{LF_1}{\varepsilon^{0.5+r}P}\end{aligned}\tag{7}$$

where ε, a dimensionless small number, denotes the small but finite transverse deformation of beams. Dimensionless parameter k_0 associated with k_1 accounts for the effects of shear distortion, parameter k_2 represents the effect of nonlinearity, parameter k_3 represents the effects of rotary inertia, and parameter k_4 denotes the stiffness of the beam. Integer r indicates the order of the external excitation. For weak external excitations, F_1 is of order εy, and $r=1$; for strong external excitations, F_1 is of order y, and $r=0$.

Substituting Eq. (7) into Eqs. (5) and (6) yields the dimensionless form:

$$\begin{aligned}&y_{,tt}+2vy_{,xt}+(v^2-1)y_{,xx}-(k_0+k_1+k_3-k_0v^2)y_{,xxtt}+(k_1+k_4-k_1v^2)y_{,xxxx}\\&\quad+k_0(y_{,tttt}+2vy_{,xttt})-2k_1vy_{,xxxt}\\&=\frac{3}{2}\varepsilon\left[k_2y_{,x}^2y_{,xx}+k_1(y_{,x}^2y_{,xxtt}+4y_{,x}y_{,xt}y_{,xxt}+2y_{,x}y_{,xx}y_{,xtt}+2y_{,xt}^2y_{,xx})\right.\\&\quad\left.-k_1k_2(y_{,x}^2y_{,xxxx}+6y_{,x}y_{,xx}y_{,xxx}+2y_{,xx}^3)\right]+\varepsilon^r b\cos(\omega T_0)\end{aligned}\tag{8}$$

$$y\big|_{x=0}=0,\quad y\big|_{x=1}=0;\quad y_{,xx}\big|_{x=0}=0,\quad y_{,xx}\big|_{x=1}=0\tag{9}$$

Lee and Jang [7] proposed a set of governing equations for axially moving Euler–Bernoulli beams to account for the momentum transport through both of the end boundaries. For the transverse vibration, their equation differs from the traditional one with an additional $y_{,xxtt}$ term. However, the present investigation still adopts the traditional formulation. It should be noticed that Eq. (8) contains term $y_{,xxtt}$. Therefore, the following analysis may be valid for models developed by Lee and Jang's approach.

3. The multi-scale analysis

The method of multiple scales will be employed to Eq. (8). One assumes an expansion of dimensionless displacement and its time derivatives

$$y(x,t;\varepsilon)=y_0(x,T_0,T_1,T_2)+\varepsilon y_1(x,T_0,T_1,T_2)+\varepsilon^2 y_2(x,T_0,T_1,T_2)\cdots \tag{10}$$

$$\begin{aligned}
\frac{d}{dt}&=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\varepsilon^2\frac{\partial}{\partial T_2}+\cdots,\\
\frac{d^2}{dt^2}&=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1}+\varepsilon^2\left(\frac{\partial^2}{\partial T_1^2}+2\frac{\partial^2}{\partial T_0\partial T_2}\right)+\cdots,\\
\frac{d^3}{dt^3}&=\frac{\partial^3}{\partial T_0^3}+3\varepsilon\frac{\partial^3}{\partial T_0^2\partial T_1}+3\varepsilon^2\left(\frac{\partial^3}{\partial T_0\partial T_1^2}+\frac{\partial^3}{\partial T_0^2\partial T_2}\right)+\cdots,\\
\frac{d^4}{dt^4}&=\frac{\partial^4}{\partial T_0^4}+4\varepsilon\frac{\partial^4}{\partial T_0^3\partial T_1}+\varepsilon^2\left(6\frac{\partial^4}{\partial T_0^2\partial T_1^2}+4\frac{\partial^4}{\partial T_0^3\partial T_2}\right)+\cdots
\end{aligned}\tag{11}$$

3.1. Nonlinear vibrations under external excitations

First, we consider nonlinear vibrations under weak external excitations of axially moving beams on simple supports based on the Timoshenko beam model. At $r=1$, substituting Eq. (10) into Eq. (8), and then equalizing the coefficients ε^0 and ε^1 in the resulting equation, one obtains

$$\varepsilon^0:\quad \begin{aligned}&\frac{\partial^2 y_0}{\partial T_0^2}+k_0\frac{\partial^4 y_0}{\partial T_0^4}+2v\frac{\partial^2 y_0}{\partial x\partial T_0}+2k_0v\frac{\partial^4 y_0}{\partial x\partial T_0^3}+(v^2-1)\frac{\partial^2 y_0}{\partial x^2}-2k_1v\frac{\partial^4 y_0}{\partial x^3\partial T_0}\\
&-(k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_0}{\partial x^2\partial T_0^2}+(k_1+k_4-k_1v^2)\frac{\partial^4 y_0}{\partial x^4}=0\end{aligned}\tag{12}$$

$$\varepsilon^1:\quad \begin{aligned}&\frac{\partial^2 y_1}{\partial T_0^2}+k_0\frac{\partial^4 y_1}{\partial T_0^4}+2v\frac{\partial^2 y_1}{\partial x\partial T_0}+2k_0v\frac{\partial^4 y_1}{\partial x\partial T_0^3}+(v^2-1)\frac{\partial^2 y_1}{\partial x^2}-2k_1v\frac{\partial^4 y_1}{\partial x^3\partial T_0}\\
&-(k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_1}{\partial x^2\partial T_0^2}+(k_1+k_4-k_1v^2)\frac{\partial^4 y_1}{\partial x^4}\\
&=-2\frac{\partial^2 y_0}{\partial T_0\partial T_1}-4k_0\frac{\partial^4 y_0}{\partial T_0^3\partial T_1}-2v\frac{\partial^2 y_0}{\partial x\partial T_1}-6k_0v\frac{\partial^4 y_0}{\partial x\partial T_0^2\partial T_1}+\frac{3}{2}k_2\left(\frac{\partial y_0}{\partial x}\right)^2\frac{\partial^2 y_0}{\partial x^2}\\
&+3k_1\left(\frac{\partial^2 y_0}{\partial x\partial T_0}\right)^2\frac{\partial^2 y_0}{\partial x^2}+3k_1\frac{\partial y_0}{\partial x}\frac{\partial^3 y_0}{\partial x\partial T_0^2}\frac{\partial^2 y_0}{\partial x^2}-3k_1k_2\left(\frac{\partial^2 y_0}{\partial x^2}\right)^3\\
&+6k_1\frac{\partial y_0}{\partial x}\frac{\partial^2 y_0}{\partial x\partial T_0}\frac{\partial^3 y_0}{\partial x^2\partial T_0}+2(k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_0}{\partial x^2\partial T_0\partial T_1}\\
&+\frac{3}{2}k_1\left(\frac{\partial y_0}{\partial x}\right)^2\frac{\partial^4 y_0}{\partial x^2\partial T_0^2}-9k_1k_2\frac{\partial y_0}{\partial x}\frac{\partial^2 y_0}{\partial x^2}\frac{\partial^3 y_0}{\partial x^3}+2k_1v\frac{\partial^4 y_0}{\partial x^3\partial T_1}\\
&-\frac{3}{2}k_1k_2\left(\frac{\partial y_0}{\partial x}\right)^2\frac{\partial^4 y_0}{\partial x^4}+b\cos(\omega t)\end{aligned}\tag{13}$$

The solution to Eq. (12) can be assumed as

$$y_0=\phi_n(x)A_n(T_1,T_2)e^{i\omega_n T_0}+\bar{\phi}_n(x)\bar{A}_n(T_1,T_2)e^{-i\omega_n T_0} \tag{14}$$

where ϕ_n and ω_n denote the nth mode function and natural frequency and overbar represents complex conjugate. Substituting Eq. (14) into Eqs. (12) and (9) yields

$$\begin{aligned} &k_0\phi_n\omega_n^4 - 2\mathrm{i}k_0 v\phi_n'\omega_n^3 + \left[(k_0+k_1+k_3-k_0v^2)\phi_n'' - \phi_n\right]\omega_n^2 + 2\mathrm{i}v(\phi_n' - k_1\phi_n''')\omega_n \\ &\quad + (v^2-1)\phi_n'' + (k_1+k_4-k_1v^2)\phi_n'''' = 0 \end{aligned} \tag{15}$$

$$\phi_n(0)=0,\quad \phi_n(1)=0,\quad \phi_n''(0)=0,\quad \phi_n''(1)=0 \tag{16}$$

The solution to ordinary differential Eq. (15) can be expressed by

$$\phi_n(x) = C_{1n}(\mathrm{e}^{\mathrm{i}\beta_{1n}x} + C_{2n}\mathrm{e}^{\mathrm{i}\beta_{2n}x} + C_{3n}\mathrm{e}^{\mathrm{i}\beta_{3n}x} + C_{4n}\mathrm{e}^{\mathrm{i}\beta_{4n}x}) \tag{17}$$

Substituting Eq. (17) into Eqs. (15) and (16) yields

$$\begin{aligned} &(k_1+k_4-k_1v^2)\beta_{in}^4 - 2vk_1\omega_n\beta_{in}^3 - [(k_0+k_1+k_3+k_0v^2)\omega_n^2 + (1-v^2)]\beta_{in}^2 \\ &\quad + (2k_0v\omega_n^3 - 2v\omega_n)\beta_{in} + k_0\omega_n^4 - \omega_n^2 = 0 \end{aligned} \tag{18}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \beta_{1n}^2 & \beta_{2n}^2 & \beta_{3n}^2 & \beta_{4n}^2 \\ \mathrm{e}^{\mathrm{i}\beta_{1n}} & \mathrm{e}^{\mathrm{i}\beta_{2n}} & \mathrm{e}^{\mathrm{i}\beta_{3n}} & \mathrm{e}^{\mathrm{i}\beta_{4n}} \\ \beta_{1n}^2\mathrm{e}^{\mathrm{i}\beta_{1n}} & \beta_{2n}^2\mathrm{e}^{\mathrm{i}\beta_{2n}} & \beta_{3n}^2\mathrm{e}^{\mathrm{i}\beta_{3n}} & \beta_{4n}^2\mathrm{e}^{\mathrm{i}\beta_{4n}} \end{pmatrix} \begin{pmatrix} 1 \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{pmatrix} C_{1n} = 0 \tag{19}$$

For the nontrivial solution of Eq. (19), the determinant of the coefficient matrix must be zero:

$$\begin{aligned} &\left[\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{2n})} + \mathrm{e}^{\mathrm{i}(\beta_{3n}+\beta_{4n})}\right](\beta_{1n}^2-\beta_{2n}^2)(\beta_{3n}^2-\beta_{4n}^2) \\ &\quad + \left[\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{2n}+\beta_{4n})}\right](\beta_{3n}^2-\beta_{1n}^2)(\beta_{2n}^2-\beta_{4n}^2) \\ &\quad + \left[\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{4n})}\right](\beta_{2n}^2-\beta_{3n}^2)(\beta_{1n}^2-\beta_{4n}^2) = 0 \end{aligned} \tag{20}$$

Using Eqs. (19) and (20), one can obtain the modal function of the simply supported beam as follows:

$$\begin{aligned} \phi(x) = c_1\Bigg\{ &\mathrm{e}^{\mathrm{i}\beta_{1n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{2n}})}\mathrm{e}^{\mathrm{i}\beta_{2n}x} - \frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{3n}})}\mathrm{e}^{\mathrm{i}\beta_{3}x} \\ &- \left(1 - \frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(\mathrm{e}^{\mathrm{i}\beta_{3n}}-\mathrm{e}^{\mathrm{i}\beta_{2n}})} + \frac{(\beta_{4n}^2-\beta_{1n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(\mathrm{e}^{\mathrm{i}\beta_{2n}}-\mathrm{e}^{\mathrm{i}\beta_{3n}})}\right)\mathrm{e}^{\mathrm{i}\beta_{4n}x}\Bigg\} \end{aligned} \tag{21}$$

The nth eigenvalues β_{jn} ($j=1$, 2, 3, 4) and the corresponding natural frequency can be calculated numerically considering boundary conditions (9). Fig. 1 presents the natural frequencies for the first and second modes. It has been found that the natural frequencies decrease with increasing axial speed. The exact value at which the first natural frequency vanishes is called the critical speed and afterwards the system is unstable about the zero equilibrium.

If the axial speed variation frequency approaches any natural frequency of the system (12), primary response may occur. A detuning parameter σ is introduced to quantify the deviation of ω from ω_n and ω is described by

$$\omega = \omega_n + \varepsilon\sigma \tag{22}$$

where ω_n denote the nth natural frequency of free vibration described by Eq. (12). To investigate the primary response with the possible contributions of modes not involved in the resonance, the solution to Eq. (13) can be expressed as

$$y_0 = \phi_n(x)A_n(T_1,T_2)\mathrm{e}^{\mathrm{i}\omega_n T_0} + \phi_m(x)A_m(T_1,T_2)\mathrm{e}^{\mathrm{i}\omega_m T_0} + \mathrm{cc} \tag{23}$$

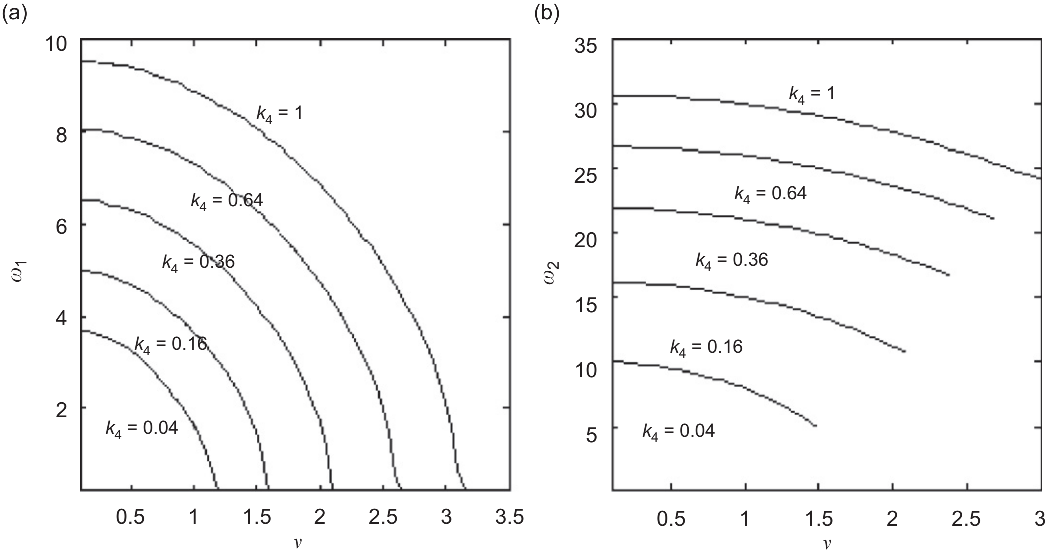

Fig. 1. Natural frequency diagram: (a) the first mode and (b) the second mode.

where $m \neq n$. Substituting Eqs. (22) and (23) into Eq. (13) yields

$$
\begin{aligned}
&\frac{\partial^2 y_1}{\partial T_0^2} + k_0\frac{\partial^4 y_1}{\partial T_0^4} + 2v\frac{\partial^2 y_1}{\partial x\,\partial T_0} + 2k_0 v\frac{\partial^4 y_1}{\partial x\,\partial T_0^3} + (v^2-1)\frac{\partial^2 y_1}{\partial x^2} - 2k_1 v \frac{\partial^4 y_1}{\partial x^3\,\partial T_0} \\
&\quad - (k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_1}{\partial x^2\,\partial T_0^2} + (k_1+k_4-k_1v^2)\frac{\partial^4 y_1}{\partial x^4} \\
&= \Big[-2(\mathrm{i}\omega_n\phi_n - 2\mathrm{i}k_0\omega_n^3\phi_n + v\phi_n' - 3k_0 v\omega_n^2\phi_n' - \mathrm{i}(k_0+k_1+k_3-k_0v^2)\omega_n\phi_n'' - k_1 v\phi_n''')\frac{\partial A_n}{\partial T_1} \\
&\quad + \frac{3}{2}(2k_2\phi_n'\bar\phi_n'\phi_n'' - 2k_1\omega_n^2\phi_n'\bar\phi_n'\phi_n'' + k_2\phi'^2_n\bar\phi_n'' - k_1\omega_n^2\phi'^2_n\bar\phi''_n - 6k_1k_2\phi''^2_n\bar\phi''_n - 6k_1k_2\bar\phi'_n\phi''_n\phi'''_n \\
&\quad - 6k_1k_2\phi'_n\bar\phi''_n\phi'''_n - 6k_1k_2\phi'_n\phi''_n\bar\phi'''_n - 2k_1k_2\phi'_n\bar\phi'_n\phi''''_n - k_1k_2\phi'^2_n\bar\phi''''_n)A_n^2\bar A_n + 3(k_2\phi'_n\bar\phi'_m\phi''_m - k_1\omega_n^2\phi'_n\bar\phi'_m\phi''_m \\
&\quad + k_2\phi'_m\bar\phi'_m\phi''_n - k_1\omega_n^2\phi'_m\bar\phi'_m\phi''_n + k_2\phi'_m\phi'_n\bar\phi''_m - k_1\omega_n^2\phi'_m\phi'_n\bar\phi''_m - 6k_1k_2\phi''_m\phi''_n\bar\phi''_m - 3k_1k_2\bar\phi'_m\phi''_n\phi'''_m \\
&\quad - 3k_1k_2\phi'_n\bar\phi''_m\phi'''_m - 3k_1k_2\bar\phi'_m\phi''_m\phi'''_n - 3k_1k_2\phi'_m\bar\phi''_m\phi'''_n - 3k_1k_2\phi'_n\phi''_m\bar\phi'''_m - 3k_1k_2\phi'_m\phi''_n\bar\phi'''_m - k_1k_2\phi'_n\bar\phi'_m\phi''''_m \\
&\quad - k_1k_2\phi'_m\bar\phi'_m\phi''''_n - k_1k_2\phi'_m\phi'_n\bar\phi''''_m)A_mA_n\bar A_m + \frac{1}{2} b\, \mathrm{e}^{\mathrm{i}\sigma T_1}\Big]\, \mathrm{e}^{\mathrm{i}\omega_n T_0} \\
&\quad + [-2(\mathrm{i}\omega_m\phi_m - 2\mathrm{i}k_0\omega_m^3\phi_m + v\phi'_m - 3k_0v\omega_m^2\phi'_m - \mathrm{i}(k_0+k_1+k_3-k_0v^2)\omega_m\phi''_m - k_1v\phi'''_m)\frac{\partial A_m}{\partial T_1} \\
&\quad + \frac{3}{2}(2k_2\phi'_m\bar\phi'_m\phi''_m - 2k_1\omega_m^2\phi'_m\bar\phi'_m\phi''_m + k_2\phi'^2_m\bar\phi''_m - k_1\omega_m^2\phi'^2_m\bar\phi''_m - 6k_1k_2\phi''^2_m\bar\phi''_n - 6k_1k_2\bar\phi'_m\phi''_m\phi'''_m \\
&\quad - 6k_1k_2\phi'_m\bar\phi''_m\phi'''_m - 6k_1k_2\phi'_m\phi''_m\bar\phi'''_m - 2k_1k_2\phi'_m\bar\phi'_m\phi''''_m - k_1k_2\phi'^2_m\bar\phi''''_m)A_m^2\bar A_m + 3(k_2\phi'_m\bar\phi'_n\phi''_n - k_1\omega_m^2\phi'_m\bar\phi'_n\phi''_n \\
&\quad + k_2\phi'_n\bar\phi'_n\phi''_m - k_1\omega_m^2\phi'_n\bar\phi'_n\phi''_m + k_2\phi'_n\phi'_m\bar\phi''_n - k_1\omega_m^2\phi'_n\phi'_m\bar\phi''_n - 6k_1k_2\phi''_n\phi''_m\bar\phi''_n - 3k_1k_2\bar\phi'_n\phi''_m\phi'''_n \\
&\quad - 3k_1k_2\phi'_m\bar\phi''_n\phi'''_n - 3k_1k_2\bar\phi'_n\phi''_n\phi'''_m - 3k_1k_2\phi'_n\bar\phi''_n\phi'''_m - 3k_1k_2\phi'_m\phi''_n\bar\phi'''_n - 3k_1k_2\phi'_n\phi''_m\bar\phi'''_n - k_1k_2\phi'_m\bar\phi'_n\phi''''_n \\
&\quad - k_1k_2\phi'_n\bar\phi'_n\phi''''_m - k_1k_2\phi'_n\phi'_m\bar\phi''''_n)A_nA_m\bar A_n]\, \mathrm{e}^{\mathrm{i}\omega_m T_0} + \mathrm{cc} + \mathrm{NST}
\end{aligned}
\tag{24}
$$

where the prime denotes derivation with respect to the dimensionless spatial variable x, cc stands for complex conjugate of the proceeding terms, NST for non-secular terms, and h.o.t. for high orders of ε.

The solvability condition, which demands orthogonal relationships, has been presented by Nayfeh [19]:

$$\begin{aligned}
&\Big\langle\Big[-2(\mathrm{i}\omega_n\phi_n - 2\mathrm{i}k_0\omega_n^3\phi_n + v\phi_n' - 3k_0v\omega_n^2\phi_n' - \mathrm{i}(k_0+k_1+k_3-k_0v^2)\omega_n\phi_n'' - k_1v\phi_n''')\frac{\partial A_n}{\partial T_1} \\
&+\frac{3}{2}(2k_2\phi_n'\bar{\phi}_n'\phi_n'' - 2k_1\omega_n^2\phi_n'\bar{\phi}_n'\phi_n'' + k_2\phi'^2_n\bar{\phi}_n'' - k_1\omega_n^2\phi'^2_n\bar{\phi}_n'' - 6k_1k_2\phi''^2_n\bar{\phi}_n'' - 6k_1k_2\bar{\phi}_n'\phi_n''\phi_n''' \\
&- 6k_1k_2\phi_n'\bar{\phi}''_n\phi_n''' - 6k_1k_2\phi_n'\phi_n''\bar{\phi}'''_n - 2k_1k_2\phi_n'\bar{\phi}_n'\phi_n'''' - k_1k_2\phi'^2_n\bar{\phi}_n'''')A_n^2\bar{A}_n \\
&+3(k_2\phi_n'\bar{\phi}_m'\phi_m'' - k_1\omega_n^2\phi_n'\bar{\phi}_m'\phi_m'' + k_2\phi_m'\bar{\phi}_m'\phi_n'' - k_1\omega_n^2\phi_m'\bar{\phi}_m'\phi_n'' + k_2\phi_m'\phi_n'\bar{\phi}_m'' - k_1\omega_n^2\phi_m'\phi_n'\bar{\phi}_m'' \\
&- 6k_1k_2\phi_m''\phi_n''\bar{\phi}_m'' - 3k_1k_2\bar{\phi}_m'\phi_n''\phi_m''' - 3k_1k_2\phi_n'\bar{\phi}_m''\phi_m''' - 3k_1k_2\bar{\phi}_m'\phi_m''\phi_n''' - 3k_1k_2\phi_m'\bar{\phi}_m''\phi_n''' \\
&- 3k_1k_2\phi_n'\phi_m''\bar{\phi}'''_m - 3k_1k_2\phi_m'\phi_n''\bar{\phi}'''_m - k_1k_2\phi_n'\bar{\phi}_m'\phi_m'''' - k_1k_2\phi_m'\bar{\phi}_m'\phi_n'''' - k_1k_2\phi_m'\phi_n'\bar{\phi}_m'''')|A_m|^2A_n \\
&+\frac{1}{2}b\,\mathrm{e}^{\mathrm{i}\sigma T_1}\Big],\phi_n\Big\rangle = 0
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\Big\langle\Big[-2(\mathrm{i}\omega_m\phi_m - 2\mathrm{i}k_0\omega_m^3\phi_m + v\phi_m' - 3k_0v\omega_m^2\phi_m' - \mathrm{i}(k_0+k_1+k_3-k_0v^2)\omega_m\phi_m'' - k_1v\phi_m''')\frac{\partial A_m}{\partial T_1} \\
&+\frac{3}{2}(2k_2\phi_m'\bar{\phi}_m'\phi_m'' - 2k_1\omega_m^2\phi_m'\bar{\phi}_m'\phi_m'' + k_2\phi'^2_m\bar{\phi}_m'' - k_1\omega_m^2\phi'^2_m\bar{\phi}_m'' - 6k_1k_2\phi''^2_m\bar{\phi}_n'' - 6k_1k_2\bar{\phi}_m'\phi_m''\phi_m''' \\
&- 6k_1k_2\phi_m'\bar{\phi}_m''\phi_m''' - 6k_1k_2\phi_m'\phi_m''\bar{\phi}'''_m - 2k_1k_2\phi_m'\bar{\phi}_m'\phi_m'''' - k_1k_2\phi'^2_m\bar{\phi}_m'''')A_m^2\bar{A}_m + 3(k_2\phi_m'\bar{\phi}_n'\phi_n'' \\
&- k_1\omega_m^2\phi_m'\bar{\phi}_n'\phi_n'' + k_2\phi_n'\bar{\phi}_n'\phi_m'' - k_1\omega_m^2\phi_n'\bar{\phi}_n'\phi_m'' + k_2\phi_n'\phi_m'\bar{\phi}_n'' - k_1\omega_m^2\phi_n'\phi_m'\bar{\phi}_n'' - 6k_1k_2\phi_n''\phi_m''\bar{\phi}_n'' \\
&- 3k_1k_2\bar{\phi}_n'\phi_m''\phi_n''' - 3k_1k_2\phi_m'\bar{\phi}_n''\phi_n''' - 3k_1k_2\bar{\phi}_n'\phi_n''\phi_m''' - 3k_1k_2\phi_n'\bar{\phi}_n''\phi_m''' - 3k_1k_2\phi_m'\phi_n''\bar{\phi}_n''' \\
&-3k_1k_2\phi_n'\phi_m''\bar{\phi}'''_n - k_1k_2\phi_m'\bar{\phi}_n'\phi_n'''' - k_1k_2\phi_n'\bar{\phi}_n'\phi_m'''' - k_1k_2\phi_n'\phi_m'\bar{\phi}_n'''')|A_n|^2A_m\Big],\phi_m\Big\rangle = 0
\end{aligned} \tag{26}$$

where the inner product is defined for complex functions on [0,1] as

$$\langle f,g\rangle = \int_0^1 f\bar{g}\,\mathrm{d}x \tag{27}$$

Application of the distributive law of the inner product to Eqs. (25) and (26) leads to

$$\frac{\partial A_n}{\partial T_1} + \kappa_nA_n^2\bar{A}_n + \mu_{nm}|A_m|^2A_n + b\chi_n\,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0 \tag{28}$$

$$\frac{\partial A_m}{\partial T_1} + \kappa_mA_m^2\bar{A}_m + \mu_{mn}|A_n|^2A_m = 0 \tag{29}$$

where

$$\begin{aligned}
\kappa_n = &-\frac{3}{2}\Big(2k_2\int_0^1\phi_n'\bar{\phi}_n'\phi_n''\bar{\phi}_n\,\mathrm{d}x - 2k_1\omega_n^2\int_0^1\phi_n'\bar{\phi}_n'\phi_n''\bar{\phi}_n\,\mathrm{d}x + k_2\int_0^1\phi'^2_n\bar{\phi}''_n\bar{\phi}_n\,\mathrm{d}x - k_1\omega_n^2\int_0^1\phi'^2_n\bar{\phi}''_n\bar{\phi}_n\,\mathrm{d}x \\
&-6k_1k_2\int_0^1\phi''^2_n\bar{\phi}''_n\bar{\phi}_n\,\mathrm{d}x - 6k_1k_2\int_0^1\bar{\phi}'_n\phi_n''\phi_n'''\bar{\phi}_n\,\mathrm{d}x - 6k_1k_2\int_0^1\phi_n'\bar{\phi}''_n\phi_n'''\bar{\phi}_n\,\mathrm{d}x - 6k_1k_2\int_0^1\phi_n'\phi_n''\bar{\phi}'''_n\bar{\phi}_n\,\mathrm{d}x \\
&-2k_1k_2\int_0^1\phi_n'\bar{\phi}'_n\phi_n''''\bar{\phi}_n\,\mathrm{d}x - k_1k_2\int_0^1\phi'^2_n\bar{\phi}_n''''\bar{\phi}_n\,\mathrm{d}x\Big)\Big/2\Big(\mathrm{i}\omega_n\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x - 2\mathrm{i}k_0\omega_n^3\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x \\
&+v\int_0^1\phi_n'\bar{\phi}_n\,\mathrm{d}x - 3k_0v\omega_n^2\int_0^1\phi_n'\bar{\phi}_n\,\mathrm{d}x - \mathrm{i}(k_0+k_1+k_3-k_0v^2)\omega_n\int_0^1\phi_n''\bar{\phi}_n\,\mathrm{d}x - k_1v\int_0^1\phi_n'''\bar{\phi}_n\,\mathrm{d}x\Big)
\end{aligned} \tag{30a}$$

$$\mu_{nm} = -3\Big(k_2\int_0^1 \phi'_n\bar{\phi}'_m\phi''_m\bar{\phi}_n\,\mathrm{d}x - k_1\omega_n^2\int_0^1 \phi'_n\bar{\phi}'_m\phi''_m\bar{\phi}_n\,\mathrm{d}x + k_2\int_0^1 \phi'_m\bar{\phi}'_m\phi''_n\bar{\phi}_n\,\mathrm{d}x$$
$$- k_1\omega_n^2\int_0^1 \phi'_m\bar{\phi}'_m\phi''_n\bar{\phi}_n\,\mathrm{d}x + k_2\int_0^1 \phi'_m\phi'_n\bar{\phi}''_m\bar{\phi}_n\,\mathrm{d}x - k_1\omega_n^2\int_0^1 \phi'_m\phi'_n\bar{\phi}''_m\bar{\phi}_n\,\mathrm{d}x$$
$$- 6k_1k_2\int_0^1 \phi''_m\phi''_n\bar{\phi}''_m\bar{\phi}_n\,\mathrm{d}x - 3k_1k_2\int_0^1 \bar{\phi}'_m\phi''_n\phi'''_m\bar{\phi}_n\,\mathrm{d}x - 3k_1k_2\int_0^1 \phi'_n\bar{\phi}''_m\phi'''_m\bar{\phi}_n\,\mathrm{d}x$$
$$- 3k_1k_2\int_0^1 \bar{\phi}'_m\phi''_m\phi'''_n\bar{\phi}_n\,\mathrm{d}x - 3k_1k_2\int_0^1 \phi'_m\bar{\phi}''_m\phi'''_n\bar{\phi}_n\,\mathrm{d}x - 3k_1k_2\int_0^1 \phi'_n\phi''_m\bar{\phi}'''_m\bar{\phi}_n\,\mathrm{d}x$$
$$- 3k_1k_2\int_0^1 \phi'_m\phi''_n\bar{\phi}'''_m\bar{\phi}_n\,\mathrm{d}x - k_1k_2\int_0^1 \phi'_n\bar{\phi}'_m\phi''''_m\bar{\phi}_n\,\mathrm{d}x - k_1k_2\int_0^1 \phi'_m\bar{\phi}'_m\phi''''_n\bar{\phi}_n\,\mathrm{d}x$$
$$-k_1k_2\int_0^1 \phi'_m\phi'_n\bar{\phi}''''_m\bar{\phi}_n\,\mathrm{d}x\Big)\Big/2\Big(\mathrm{i}\omega_n\int_0^1 \phi_n\bar{\phi}_n\,\mathrm{d}x - 2\mathrm{i}k_0\omega_n^3\int_0^1 \phi_n\bar{\phi}_n\,\mathrm{d}x + v\int_0^1 \phi'_n\bar{\phi}_n\,\mathrm{d}x$$
$$-3k_0v\omega_n^2\int_0^1 \phi'_n\bar{\phi}_n\,\mathrm{d}x - \mathrm{i}(k_0+k_1+k_3-k_0v^2)\omega_n\int_0^1 \phi''_n\bar{\phi}_n\,\mathrm{d}x - k_1v\int_0^1 \phi'''_n\bar{\phi}_n\,\mathrm{d}x\Big) \tag{30b}$$

$$\chi_n = -\int_0^1 \bar{\phi}_n\,\mathrm{d}x\Big/4\Big(\mathrm{i}\omega_n\int_0^1 \phi_n\bar{\phi}_n\,\mathrm{d}x - 2\mathrm{i}k_0\omega_n^3\int_0^1 \phi_n\bar{\phi}_n\,\mathrm{d}x + v\int_0^1 \phi'_n\bar{\phi}_n\,\mathrm{d}x - 3k_0v\omega_n^2\int_0^1 \phi'_n\bar{\phi}_n\,\mathrm{d}x$$
$$-\mathrm{i}(k_0+k_1+k_3-k_0v^2)\omega_n\int_0^1 \phi''_n\bar{\phi}_n\,\mathrm{d}x - k_1v\int_0^1 \phi'''_n\bar{\phi}_n\,\mathrm{d}x\Big) \tag{30c}$$

Express the solution to Eqs. (28) and (29) in polar form

$$A_n = \alpha_n(T_1,T_2)\,\mathrm{e}^{\mathrm{i}\beta_n(T_1,T_2)},$$
$$A_m = \alpha_m(T_1,T_2)\,\mathrm{e}^{\mathrm{i}\beta_m(T_1,T_2)} \tag{31}$$

Substituting Eq. (31) into Eq. (29) yields

$$\frac{\partial\alpha_m}{\partial T_1} = -\mathrm{Re}(\kappa_m)\beta_m^3 - \mathrm{Re}(\mu_{mn})|A_n|^2\beta_m,$$
$$\alpha_m\frac{\partial\beta_m}{\partial T_1} = -\mathrm{Im}(\kappa_m)\beta_m^3 - \mathrm{Im}(\mu_{mn})|A_n|^2\beta_m \tag{32}$$

For steady-state solutions, the amplitude α_m and the new phase β_m angle in Eq. (32) should be constant. Setting $\alpha'_m = 0$ and $\beta'_m = 0$ gives

$$0 = -\mathrm{Re}(\kappa_m)\beta_m^3 - \mathrm{Re}(\mu_{mn})|A_n|^2\beta_m,$$
$$0 = -\mathrm{Im}(\kappa_m)\beta_m^3 - \mathrm{Im}(\mu_{mn})|A_n|^2\beta_m \tag{33}$$

Eq. (33) obviously has a zero solution. If we assume that there is a non-zero solution, Eq. (33) yields

$$0 = -\mathrm{Re}(\kappa_m)\beta_m^2 - \mathrm{Re}(\mu_{mn})|A_n|^2,$$
$$0 = -\mathrm{Im}(\kappa_m)\beta_m^2 - \mathrm{Im}(\mu_{mn})|A_n|^2 \tag{34}$$

The two equations in Eq. (34) cannot come into existence at the same time. Thus, the assumption is wrong. The solution to Eq. (29) has only zero stationary solution and decays to zero exponentially. Therefore, the mth mode has actually no effect on the stability.

Substituting Eq. (31) into Eq. (28) and eliminating the mth mode yields

$$\frac{\partial\alpha_n}{\partial T_1} = -b\,\mathrm{Re}(\chi_n)\cos\theta_n + b\,\mathrm{Im}(\chi_n)\sin\theta_n,$$
$$\alpha_n\frac{\partial\theta_n}{\partial T_1} = \alpha_n\sigma + \mathrm{Im}(\kappa_n)\alpha_n^3 + b\,\mathrm{Re}(\chi_n)\sin\theta_n + b\,\mathrm{Im}(\chi_n)\cos\theta_n \tag{35}$$

where

$$\theta_n = \sigma T_1 - \beta_n \tag{36}$$

For steady-state solutions, the amplitude α_n and the new phase θ_n angle in Eq. (35) should be constant. Setting $\alpha_n' = 0$ and $\theta_n' = 0$ leads to

$$\sigma = -\mathrm{Im}(\kappa_n)\alpha_n^2 \pm \frac{b}{\alpha_n}|\chi_n| \tag{37}$$

$\alpha_n \neq 0$ in Eq. (35) yields

$$\begin{aligned} \frac{\partial \alpha_n}{\partial T_1} &= -b\,\mathrm{Re}(\chi_n)\cos\theta_n + b\,\mathrm{Im}(\chi_n)\sin\theta_n, \\ \frac{\partial \theta_n}{\partial T_1} &= \sigma + \mathrm{Im}(\kappa_n)\alpha_n^2 + \frac{b}{\alpha_n}\mathrm{Re}(\chi_n)\ \sin\ \theta_n + \frac{b}{\alpha_n}\mathrm{Im}(\chi_n)\ \cos\ \theta_n \end{aligned} \tag{38}$$

The stability of the nontrivial state can be obtained by perturbing these polar modulation equations and checking the eigenvalues of the resulting Jacobian matrix.

To determine the stability of the nontrivial state, these equations are perturbed to obtain

$$\left\{\Delta\frac{\partial \alpha_n}{\partial T_1}, \Delta\frac{\partial \theta_n}{\partial T_1}\right\}^{\mathrm{T}} = \mathbf{J}\{\Delta\alpha_n, \Delta\theta_n\}^{\mathrm{T}} \tag{39}$$

where T denotes transpose.

The Jacobian matrix whose eigenvalues determine the stability and bifurcations of the system is

$$\mathbf{J} = \begin{pmatrix} 0 & -\sigma\alpha_n - \mathrm{Im}(\kappa_n)\alpha_n^3 \\ \dfrac{\sigma}{\alpha_n} + 3\,\mathrm{Im}(\kappa_n)\alpha_n & 0 \end{pmatrix} \tag{40}$$

The **J**-matrix characteristic function

$$\lambda^2 + \left[\sigma + \mathrm{Im}(\kappa_n)\alpha_n^2\right]\left[\sigma + 3\mathrm{Im}(\kappa_n)\alpha_n^2\right] = 0 \tag{41}$$

where λ is an eigenvalue of the system. Based on Routh–Hurwitz theorem, the boundary of instability yields

$$\left[\sigma + \mathrm{Im}(\kappa_n)\alpha_n^2\right]\left[\sigma + 3\,\mathrm{Im}(\kappa_n)\alpha_n^2\right] = 0 \tag{42}$$

Consider an axially moving beam with $k_4 = 0.64$ and $v = 2.0$. The response amplitudes at exact resonance for the first two natural frequencies are shown in Fig. 2. The first two natural frequencies of the unperturbed system are $\omega_1 = 4.7393$ and $\omega_2 = 23.7017$. In the first-mode response, the coefficients are $b = 0.04$, $k_2 = 100{,}000$, and in the second-mode response, the coefficients are $b = 0.1$, $k_2 = 100{,}000$.

Fig. 3 shows the effects of a different parameter k_2. With an increase of k_2, response under the same conditions decreases. In the first-mode response, the coefficient is $b = 0.04$, and in the second-mode response, the coefficient is $b = 0.1$. The solid lines are for coefficient $k_2 = 50{,}000$, the dashed for coefficient $k_2 = 100{,}000$, and the dotted lines for coefficient $k_2 = 150{,}000$.

The effects of the foundation vibration amplitude on the response amplitudes are illustrated in Fig. 4. From the response diagrams, it is clear that the excitation amplitudes increase the amplitude of the excited system. The coefficient is $k_2 = 100{,}000$. In the first-mode response, the solid lines are for coefficient $b = 0.02$, the dashed for coefficient $b = 0.04$, and the dotted lines for coefficient $b = 0.06$. In the second-mode response, the solid lines are for coefficient $b = 0.05$, the dashed are for coefficient $b = 0.1$, and the dotted lines for coefficient $b = 0.15$.

The stability of the response amplitudes is illustrated in Fig. 5. In the first-mode response, the coefficients are $b = 0.04$ and $k_2 = 100{,}000$. In the second-mode response, the coefficients are $b = 0.1$ and $k_2 = 100{,}000$. The solid lines denote the response amplitudes and the dashed denote the boundary of instability.

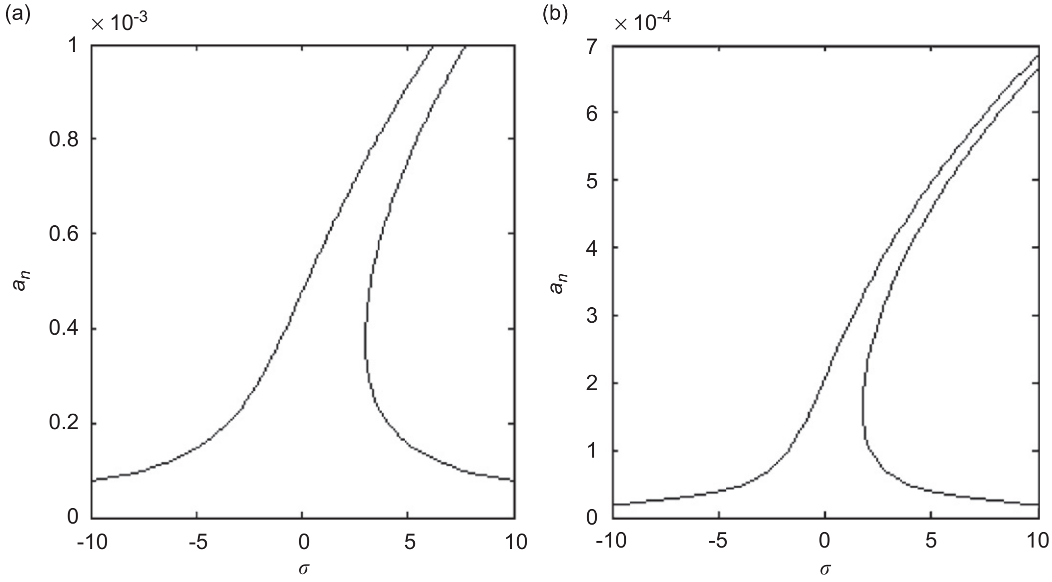

Fig. 2. Response amplitudes diagram: (a) the first mode and (b) the second mode.

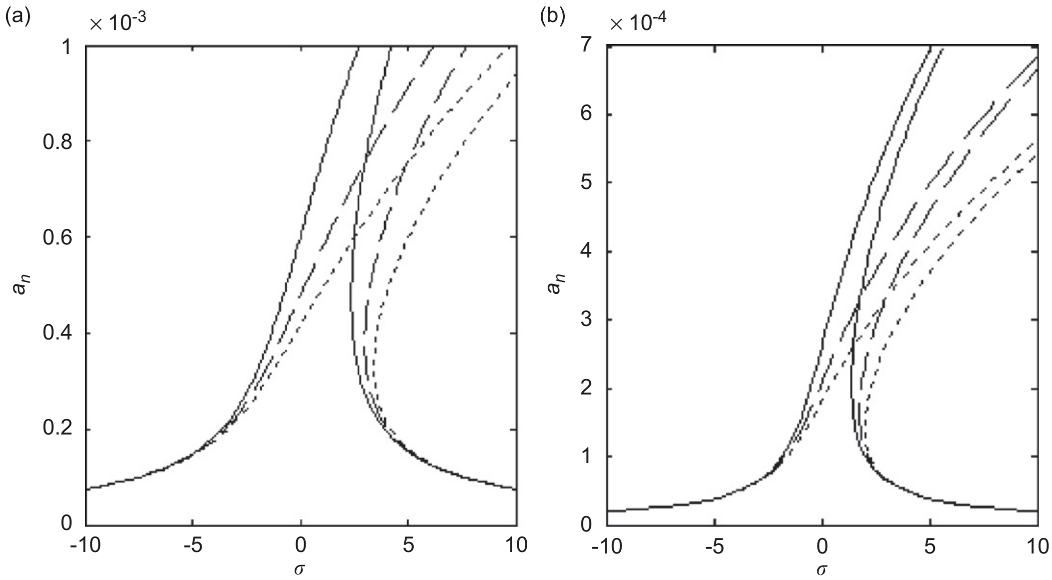

Fig. 3. Comparison of response amplitudes of a different parameter k_2: (a) the first mode and (b) the second mode.

The inner region of the boundary of instability is unstable and the outer is stable. Thus a jump phenomenon appears.

3.2. Nonlinear vibrations under strong external excitations

Next, we consider nonlinear vibrations under strong external excitations of axially moving beams on simple supports based on the Timoshenko beam model. At $r = 0$, substituting Eq. (10) into Eq. (8), and then

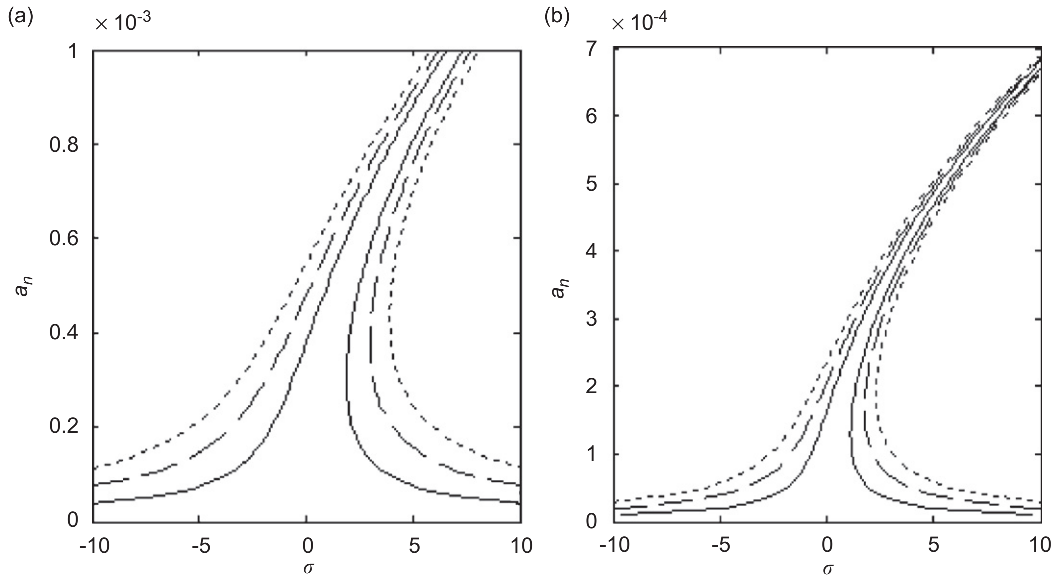

Fig. 4. Comparison of response amplitudes of different excitation amplitudes b: (a) the first mode and (b) the second mode.

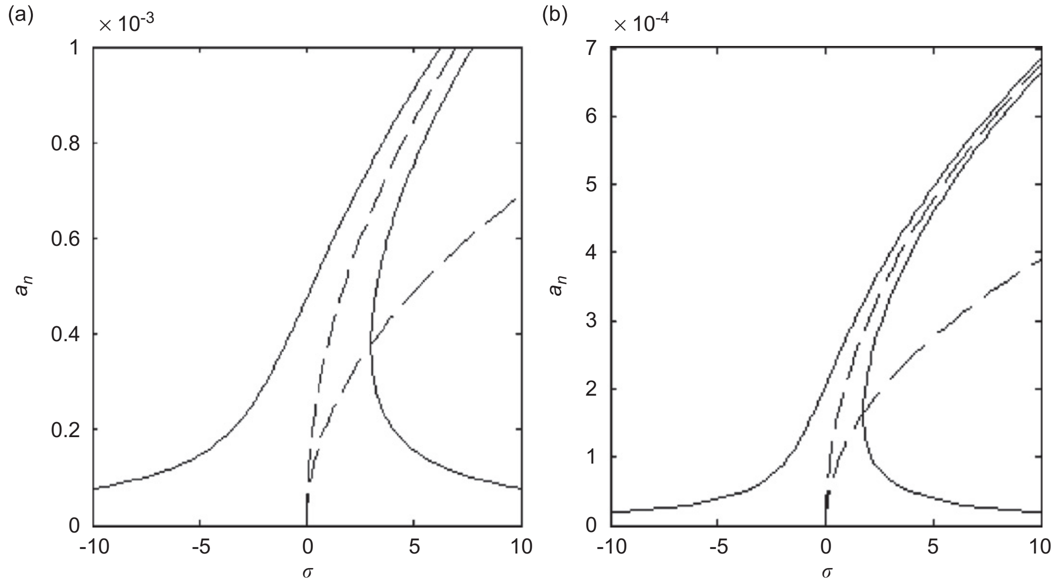

Fig. 5. Stability of the response amplitudes: (a) the first mode and (b) the second mode.

equalizing the coefficients ε^0 and ε^1 in the resulting equation, one obtains

$$\varepsilon^0: \quad \begin{aligned} &\frac{\partial^2 y_0}{\partial T_0^2} + k_0\frac{\partial^4 y_0}{\partial T_0^4} + 2v\frac{\partial^2 y_0}{\partial x\,\partial T_0} + 2k_0v\frac{\partial^4 y_0}{\partial x\,\partial T_0^3} + (v^2-1)\frac{\partial^2 y_0}{\partial x^2} - 2k_1v\frac{\partial^4 y_0}{\partial x^3\,\partial T_0} \\ &-(k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_0}{\partial x^2\,\partial T_0^2} + (k_1+k_4-k_1v^2)\frac{\partial^4 y_0}{\partial x^4} = b\cos(\omega T_0) \end{aligned} \tag{43}$$

$$\varepsilon^1: \quad \begin{aligned} &\frac{\partial^2 y_1}{\partial T_0^2} + k_0\frac{\partial^4 y_1}{\partial T_0^4} + 2v\frac{\partial^2 y_1}{\partial x\,\partial T_0} + 2k_0 v\frac{\partial^4 y_1}{\partial x\,\partial T_0^3} + (v^2-1)\frac{\partial^2 y_1}{\partial x^2} - 2k_1 v\frac{\partial^4 y_1}{\partial x^3\,\partial T_0} \\ &\quad -(k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_1}{\partial x^2\,\partial T_0^2} + (k_1+k_4-k_1v^2)\frac{\partial^4 y_1}{\partial x^4} \\ &= -2\frac{\partial^2 y_0}{\partial T_0\,\partial T_1} - 4k_0\frac{\partial^4 y_0}{\partial T_0^3\,\partial T_1} - 2v\frac{\partial^2 y_0}{\partial x\,\partial T_1} - 6k_0 v\frac{\partial^4 y_0}{\partial x\,\partial T_0^2\,\partial T_1} + \frac{3}{2}k_2\left(\frac{\partial y_0}{\partial x}\right)^2\frac{\partial^2 y_0}{\partial x^2} \\ &\quad +3k_1\left(\frac{\partial^2 y_0}{\partial x\,\partial T_0}\right)^2\frac{\partial^2 y_0}{\partial x^2} + 3k_1\frac{\partial y_0}{\partial x}\frac{\partial^3 y_0}{\partial x\,\partial T_0^2}\frac{\partial^2 y_0}{\partial x^2} - 3k_1k_2\left(\frac{\partial^2 y_0}{\partial x^2}\right)^3 \\ &\quad +6k_1\frac{\partial y_0}{\partial x}\frac{\partial^2 y_0}{\partial x\,\partial T_0}\frac{\partial^3 y_0}{\partial x^2\,\partial T_0} + 2(k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_0}{\partial x^2\,\partial T_0\,\partial T_1} \\ &\quad +\frac{3}{2}k_1\left(\frac{\partial y_0}{\partial x}\right)^2\frac{\partial^4 y_0}{\partial x^2\,\partial T_0^2} - 9k_1k_2\frac{\partial y_0}{\partial x}\frac{\partial^2 y_0}{\partial x^2}\frac{\partial^3 y_0}{\partial x^3} + 2k_1 v\frac{\partial^4 y_0}{\partial x^3\,\partial T_1} - \frac{3}{2}k_1k_2\left(\frac{\partial y_0}{\partial x}\right)^2\frac{\partial^4 y_0}{\partial x^4} \end{aligned} \tag{44}$$

Express the solution to Eq. (43) in the following form:

$$y_0(x, T_0, T_1, T_2) = \phi(x)q(T_0, T_1, T_2) + \mathrm{cc} \tag{45}$$

where $\phi(x)$ is the complex mode function satisfying the boundary condition. The mode function has been calculated above in Eq. (21).

Substituting Eq. (45) into Eq. (43), multiplying both sides by $\phi(x)$, and integrating on [0,1] gives

$$q^{(4)} + \alpha_1\dddot{q} + \alpha_2\ddot{q} + \alpha_3\dot{q} + \alpha_4 q = h\cos(\omega T_0) \tag{46}$$

where

$$\alpha_1 = \frac{2v\int_0^1 \phi\phi'\,\mathrm{d}x}{\int_0^1 \phi^2\,\mathrm{d}x}, \quad \alpha_2 = \frac{\int_0^1 \phi^2\,\mathrm{d}x - (k_0+k_1+k_3-k_0v^2)\int_0^1 \phi\phi''\,\mathrm{d}x}{k_0\int_0^1 \phi^2\,\mathrm{d}x}, \quad h = \frac{b\int_0^1 \phi\,\mathrm{d}x}{k_0\int_0^1 \phi^2\,\mathrm{d}x},$$
$$\alpha_3 = \frac{2v\left(\int_0^1 \phi\phi'\,\mathrm{d}x - k_1\int_0^1 \phi\phi'''\,\mathrm{d}x\right)}{k_0\int_0^1 \phi^2\,\mathrm{d}x}, \quad \alpha_4 = \frac{(v^2-1)\int_0^1 \phi\phi''\,\mathrm{d}x + (k_1+k_4-k_1v^2)\int_0^1 \phi\phi''''\,\mathrm{d}x}{k_0\int_0^1 \phi^2\,\mathrm{d}x} \tag{47}$$

Express the solution to Eq. (46) in polar form

$$q = A(T_1, T_2)\,\mathrm{e}^{\mathrm{i}\omega_0 T_0} + B\,\mathrm{e}^{\mathrm{i}\omega T_0} \tag{48}$$

Substituting Eq. (48) into (46) yields

$$B = \frac{h}{2(\omega^4 - \mathrm{i}\alpha_1\omega^3 - \alpha_2\omega^2 + \mathrm{i}\alpha_3\omega + \alpha_4)} \tag{49}$$

The solution of Eq. (43) is

$$y_0(x, T_0, T_1, T_2) = \phi(x)(A(T_1, T_2)\,\mathrm{e}^{\mathrm{i}\omega_0 T_0} + B\,\mathrm{e}^{\mathrm{i}\omega T_0}) + \mathrm{cc} \tag{50}$$

Substituting Eq. (50) into Eq. (44) yields

$$\begin{aligned} &\frac{\partial^2 y_1}{\partial T_0^2} + k_0\frac{\partial^4 y_1}{\partial T_0^4} + 2v\frac{\partial^2 y_1}{\partial x\,\partial T_0} + 2k_0 v\frac{\partial^4 y_1}{\partial x\,\partial T_0^3} + (v^2-1)\frac{\partial^2 y_1}{\partial x^2} - 2k_1 v\frac{\partial^4 y_1}{\partial x^3\,\partial T_0} \\ &\quad -(k_0+k_1+k_3-k_0v^2)\frac{\partial^4 y_1}{\partial x^2\,\partial T_0^2} + (k_1+k_4-k_1v^2)\frac{\partial^4 y_1}{\partial x^4} \\ &= -\frac{3}{2}[(9k_1\omega_0^2 - k_2)\phi'^2\phi'' + 2k_1k_2\phi''^3 + 6k_1k_2\phi'\phi''\phi''' + k_1k_2\phi'^2\phi'''']A^3\,\mathrm{e}^{3\mathrm{i}\omega_0 T_0} \end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{2}[(9k_1\omega^2-k_2)\phi'^2\phi''+2k_1k_2\phi''^3+6k_1k_2\phi'\phi''\phi'''+k_1k_2\phi'^2\phi'''']B^3\,\mathrm{e}^{3\mathrm{i}\omega T_0}\\
&-\frac{9}{2}[(-k_2+k_1\omega^2+4k_1\omega\omega_0+4k_1\omega_0^2)\phi'^2\phi''+2k_1k_2\phi''^3+6k_1k_2\phi'\phi''\phi'''\\
&+k_1k_2\phi'^2\phi'''']A^2B\,\mathrm{e}^{\mathrm{i}(\omega+2\omega_0)T_0}\\
&-\frac{3}{2}[(-k_2+k_1\omega^2-4k_1\omega\omega_0+4k_1\omega_0^2)(\bar{\phi}'^2\phi''+2\phi'\bar{\phi}'\bar{\phi}'')+6k_1k_2\phi''\bar{\phi}''^2\\
&+6k_1k_2(\bar{\phi}'\bar{\phi}''\phi'''+\bar{\phi}'\phi''\bar{\phi}'''+\phi'\bar{\phi}''\bar{\phi}''')+k_1k_2(\phi'^2\phi''''+2\phi'\bar{\phi}'\bar{\phi}'''')]\bar{A}^2B\,\mathrm{e}^{\mathrm{i}(\omega-2\omega_0)T_0}\\
&-\frac{9}{2}[(-k_2+k_1\omega_0^2+4k_1\omega\omega_0+4k_1\omega^2)\phi'^2\phi''+2k_1k_2\phi''^3+6k_1k_2\phi'\phi''\phi'''\\
&+k_1k_2\phi'^2\phi'''']AB^2\,\mathrm{e}^{\mathrm{i}(\omega_0+2\omega)T_0}\\
&-\frac{3}{2}[(-k_2+k_1\omega_0^2-4k_1\omega\omega_0+4k_1\omega^2)(\bar{\phi}'^2\phi''+2\phi'\bar{\phi}'\bar{\phi}'')+6k_1k_2\phi''\bar{\phi}''^2\\
&+6k_1k_2(\bar{\phi}'\bar{\phi}''\phi'''+\bar{\phi}'\phi''\bar{\phi}'''+\phi'\bar{\phi}''\bar{\phi}''')+k_1k_2(\bar{\phi}'^2\phi''''+2\phi'\bar{\phi}'\bar{\phi}'''')]A\bar{B}^2\,\mathrm{e}^{\mathrm{i}(\omega_0-2\omega)T_0}\\
&-\{3[(k_1\omega^2-k_2)(\phi'^2\bar{\phi}''+2\phi'\bar{\phi}'\phi'')+6k_1k_2(\phi''^2\bar{\phi}''+\bar{\phi}'\phi''\phi'''+\phi'\bar{\phi}''\phi'''+\phi'\phi''\bar{\phi}''')\\
&+2k_1k_2\phi'\bar{\phi}'\phi''''+k_1k_2\phi'^2\bar{\phi}'''']AB\bar{A}\\
&+\frac{3}{2}[(k_1\omega^2-k_2)(\phi'^2\bar{\phi}''+2\phi'\bar{\phi}'\phi'')+6k_1k_2(\phi''^2\bar{\phi}''+\bar{\phi}'\phi''\phi'''+\phi'\bar{\phi}''\phi'''+\phi'\phi''\bar{\phi}''')\\
&+2k_1k_2\phi'\bar{\phi}'\phi''''+k_1k_2\phi'^2\bar{\phi}'''']B^2\bar{B}\}\,\mathrm{e}^{\mathrm{i}\omega T_0}\\
&-\left\{3[(k_1\omega_0^2-k_2)(\phi'^2\bar{\phi}''+2\phi'\bar{\phi}'\phi'')+6k_1k_2(\phi''^2\bar{\phi}''+\bar{\phi}'\phi''\phi'''+\phi'\bar{\phi}''\phi'''+\phi'\phi''\bar{\phi}''')\right.\\
&+2k_1k_2\phi'\bar{\phi}'\phi''''+k_1k_2\phi'^2\bar{\phi}'''']AB\bar{B}\\
&+\frac{3}{2}[(k_1\omega_0^2-k_2)(\phi'^2\bar{\phi}''+2\phi'\bar{\phi}'\phi'')+6k_1k_2(\phi''^2\bar{\phi}''+\bar{\phi}'\phi''\phi'''+\phi'\bar{\phi}''\phi'''+\phi'\phi''\bar{\phi}''')\\
&+2k_1k_2\phi'\bar{\phi}'\phi''''+k_1k_2\phi'^2\bar{\phi}'''']A^2\bar{A}\\
&+2[\mathrm{i}\omega_0(1-2k_0\omega_0^2)\phi+v(1-3k_0\omega_0^2)\phi'-\mathrm{i}\omega_0(k_0+k_1+k_3-k_0v^2)\phi''\\
&\left.-k_1v\phi''']\frac{\partial A}{\partial T_1}\right\}\mathrm{e}^{\mathrm{i}\omega_0 T_0}+\mathrm{cc}+\mathrm{NST}+\mathrm{h.o.t.}
\end{aligned}
\tag{51}
$$

where the prime denotes derivation with respect to dimensionless spatial variable x, cc stands for complex conjugate of the proceeding terms, NST for non-secular terms, and h.o.t. for high orders of ε.

3.2.1. Nonsyntonic excitations

When the frequency ω is far from 0, ω_0, $\omega_0/3$, and $3\omega_0$, the solvability condition demands the orthogonal relationship

$$
\begin{aligned}
&\left\langle 3[(k_1\omega_0^2-k_2)(\phi'^2\bar{\phi}''+2\phi'\bar{\phi}'\phi'')+6k_1k_2(\phi''^2\bar{\phi}''+\bar{\phi}'\phi''\phi'''+\phi'\bar{\phi}''\phi'''+\phi'\phi''\bar{\phi}''')\right.\\
&+2k_1k_2\phi'\bar{\phi}'\phi''''+k_1k_2\phi'^2\bar{\phi}'''']AB\bar{B}\\
&+\frac{3}{2}[(k_1\omega_0^2-k_2)(\phi'^2\bar{\phi}''+2\phi'\bar{\phi}'\phi'')+6k_1k_2(\phi''^2\bar{\phi}''+\bar{\phi}'\phi''\phi'''+\phi'\bar{\phi}''\phi'''+\phi'\phi''\bar{\phi}''')\\
&+2k_1k_2\phi'\bar{\phi}'\phi''''+k_1k_2\phi'^2\bar{\phi}'''']A^2\bar{A}\\
&+2[\mathrm{i}\omega_0(1-2k_0\omega_0^2)\phi+v(1-3k_0\omega_0^2)\phi'-\mathrm{i}\omega_0(k_0+k_1+k_3-k_0v^2)\phi''\\
&\left.-k_1v\phi''']\frac{\partial A}{\partial T_1},\phi\right\rangle=0
\end{aligned}
\tag{52}
$$

where the inner product is defined for complex functions on [0,1] as

$$\langle f, g\rangle = \int_0^1 f\bar{g}\,\mathrm{d}x \tag{53}$$

Application of the distributive law of inner product to Eq. (52) leads to

$$\chi AB\bar{B} + \kappa A^2\bar{A} + \frac{\partial A}{\partial T_1} = 0 \tag{54}$$

where

$$\begin{aligned}\chi = 3\Bigg[&(k_1\omega_0^2 - k_2)\left(\int_0^1 \bar{\phi}\phi'^2\bar{\phi}''\,\mathrm{d}x + 2\int_0^1 \bar{\phi}\phi'\bar{\phi}'\phi''\,\mathrm{d}x\right) + 6k_1k_2\left(\int_0^1 \bar{\phi}\phi''^2\bar{\phi}''\,\mathrm{d}x\right.\\ &\left.+\int_0^1 \bar{\phi}\bar{\phi}'\phi''\phi'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\phi'\bar{\phi}''\phi'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\phi'\phi''\bar{\phi}'''\,\mathrm{d}x\right) + 2k_1k_2\\ &\times\int_0^1 \bar{\phi}\phi'\bar{\phi}'\phi''''\,\mathrm{d}x + k_1k_2\int_0^1 \bar{\phi}\phi'^2\bar{\phi}''''\,\mathrm{d}x\Bigg]\Bigg/\Bigg\{2\Bigg[\mathrm{i}\omega_0(1-2k_0\omega_0^2)\\ &\times\int_0^1 \bar{\phi}\phi\,\mathrm{d}x + v(1-3k_0\omega_0^2)\int_0^1 \bar{\phi}\phi'\,\mathrm{d}x - \mathrm{i}\omega_0(k_0+k_1+k_3-k_0v^2)\\ &\times\int_0^1 \bar{\phi}\phi''\,\mathrm{d}x - k_1 v\int_0^1 \bar{\phi}\phi'''\,\mathrm{d}x\Bigg]\Bigg\}\end{aligned} \tag{55a}$$

$$\kappa = \frac{1}{2}\chi \tag{55b}$$

These coefficients in Eq. (55) can be determined by the natural frequencies and the modal functions calculated from Eq. (12) with the boundary condition. Consider the transformation

$$\begin{aligned}A &= a_1(T_1, T_2)\,\mathrm{e}^{\mathrm{i}a_2(T_1,T_2)},\\ B &= b_1\,\mathrm{e}^{\mathrm{i}b_2}\end{aligned} \tag{56}$$

Substituting Eq. (56) into Eq. (54) yields

$$\frac{\partial a_1}{\partial T_1} = -a_1^3\kappa^R - a_1b_1^2\chi^R \tag{57a}$$

$$\frac{\partial a_2}{\partial T_1} = -a_1^2\kappa^I - b_1^2\chi^I \tag{57b}$$

The first approximate solution is

$$y_0 = a_1\cos(\omega_0 t + a_2) + B\cos(\omega t) + O(\varepsilon) \tag{58}$$

These coefficients a_1 and a_2 can be determined from Eqs. (57a) and (57b). The amplitude of the free-vibration term will be attenuated. Steady-state response will be only forced vibration term.

3.2.2. Superharmonic resonances and response amplitudes

To study the superharmonic resonances, a detuning parameter σ is introduced to quantify the deviation and ω is described by

$$3\omega = \omega_0 + \varepsilon\sigma \tag{59}$$

Solvability requires

$$\chi AB\bar{B} + \kappa A^2\bar{A} + \zeta B^3\,\mathrm{e}^{\mathrm{i}\sigma T_1} + \frac{\partial A}{\partial T_1} = 0 \tag{60}$$

where

$$\chi = 3\Big[(k_1\omega_0^2 - k_2)\Big(\int_0^1 \bar{\phi}\phi'^2\bar{\phi}''\,\mathrm{d}x + 2\int_0^1 \bar{\phi}\phi'\bar{\phi}'\phi''\,\mathrm{d}x\Big) + 6k_1k_2\Big(\int_0^1 \bar{\phi}\phi''^2\bar{\phi}''\,\mathrm{d}x + \int_0^1 \bar{\phi}\bar{\phi}'\phi''\phi'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\phi'\bar{\phi}''\phi'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\phi'\phi''\bar{\phi}'''\,\mathrm{d}x\Big) + 2k_1k_2 \times \int_0^1 \bar{\phi}\phi'\bar{\phi}'\phi''''\,\mathrm{d}x + k_1k_2\int_0^1 \bar{\phi}\phi'^2\bar{\phi}''''\,\mathrm{d}x\Big] \Big/ \Big\{2\Big[\mathrm{i}\omega_0(1-2k_0\omega_0^2) \times \int_0^1 \bar{\phi}\phi\,\mathrm{d}x + v(1-3k_0\omega_0^2)\int_0^1 \bar{\phi}\phi'\,\mathrm{d}x - \mathrm{i}\omega_0(k_0+k_1+k_3-k_0v^2) \times \int_0^1 \bar{\phi}\phi''\,\mathrm{d}x - k_1v\int_0^1 \bar{\phi}\phi'''\,\mathrm{d}x\Big]\Big\} \tag{61a}$$

$$\kappa = \frac{1}{2}\chi \tag{61b}$$

$$\zeta = \frac{3}{2}\Big[(k_1\omega_0^2 - k_2)\int_0^1 \bar{\phi}\phi'^2\phi''\,\mathrm{d}x + 2k_1k_2\int_0^1 \bar{\phi}\phi''^3\,\mathrm{d}x + 6k_1k_2\int_0^1 \bar{\phi}\phi'\phi''\phi'''\,\mathrm{d}x + k_1k_2\int_0^1 \bar{\phi}\phi'^2\phi''''\,\mathrm{d}x\Big] \Big/ \Big\{2\Big[\mathrm{i}\omega_0(1-2k_0\omega_0^2)\int_0^1 \bar{\phi}\phi\,\mathrm{d}x + v(1-3k_0\omega_0^2)\int_0^1 \bar{\phi}\phi'\,\mathrm{d}x - \mathrm{i}\omega_0(k_0+k_1+k_3-k_0v^2)\int_0^1 \bar{\phi}\phi''\,\mathrm{d}x - k_1v\int_0^1 \bar{\phi}\phi'''\,\mathrm{d}x\Big]\Big\} \tag{61c}$$

Consider the transformation

$$A = a_1(T_1,T_2)\,\mathrm{e}^{\mathrm{i}a_2(T_1,T_2)}, \quad B = b_1\,\mathrm{e}^{\mathrm{i}b_2} \tag{62}$$

Substituting Eq. (62) into Eq. (60) yields

$$\frac{\partial a_1}{\partial T_1} = -a_1^3\kappa^R - a_1b_1^2\chi^R - b_1^3[\zeta^R\cos(3b_2+\sigma T_1 - a_2) - \zeta^I\sin(3b_2+\sigma T_1 - a_2)] \tag{63a}$$

$$a_1\frac{\partial a_2}{\partial T_1} = -a_1^3\kappa^I - a_1b_1^2\chi^I - b_1^3[\zeta^R\sin(3b_2+\sigma T_1 - a_2) + \zeta^I\cos(3b_2+\sigma T_1 - a_2)] \tag{63b}$$

Then, we change Eq. (63) into an autonomous system

$$\frac{\partial a_1}{\partial T_1} = -a_1^3\kappa^R - a_1b_1^2\chi^R - b_1^3(\zeta^R\cos\gamma - \zeta^I\sin\gamma) \tag{64a}$$

$$\frac{\partial \gamma}{\partial T_1} = \sigma + a_1^2\kappa^I + b_1^2\chi^I + \frac{b_1^3}{a_1}(\zeta^R\sin\gamma + \zeta^I\cos\gamma) \tag{64b}$$

where

$$\gamma = 3b_2 + \sigma T_1 - a_2 \tag{65}$$

For steady-state solutions, the amplitude a_1 and the new phase γ angle in Eq. (64) should be constant. Setting $a_1' = 0$ and $\gamma' = 0$ and then eliminating from Eq. (64) leads to

$$\sigma = -(a_1^2\kappa^I + b_1^2\chi^I) \pm \frac{\sqrt{b_1^6(\zeta^{I2}+\zeta^{R2}) - a_1^2(a_1^2\kappa^R + b_1^2\chi^R)^2}}{a_1} \tag{66}$$

When $a_1 \neq 0$ in Eq. (64), the Jacobian matrix characteristic function of the balanced solution on the right hand,

$$\lambda^2 + 2(2a_1^2\kappa^R + b_1^2\chi^R)\lambda + (3a_1^2\kappa^R + b_1^2\chi^R)(a_1^2\kappa^R + b_1^2\chi^R) + (\sigma + a_1^2\kappa^I + b_1^2\chi^I)(\sigma + 3a_1^2\kappa^I + b_1^2\chi^I) = 0 \tag{67}$$

where λ is an eigenvalue of the system. Based on Routh–Hurwitz theorem, the boundary of instability yields

$$(3a_1^2\kappa^R + b_1^2\chi^R)(a_1^2\kappa^R + b_1^2\chi^R) + (\sigma + a_1^2\kappa^I + b_1^2\chi^I)(\sigma + 3a_1^2\kappa^I + b_1^2\chi^I) = 0 \tag{68}$$

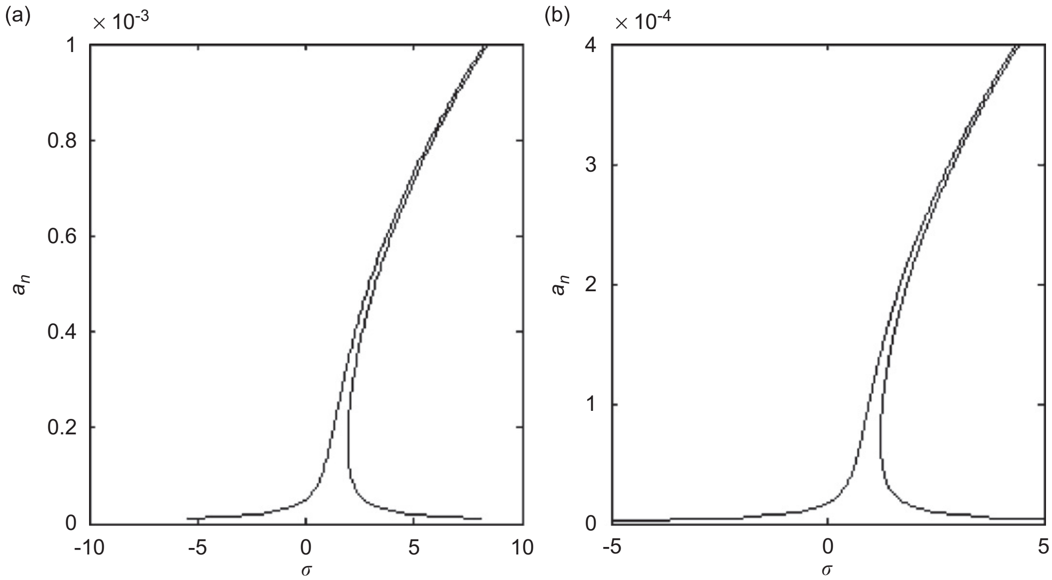

Fig. 6. Response amplitude diagram: (a) the first mode and (b) the second mode.

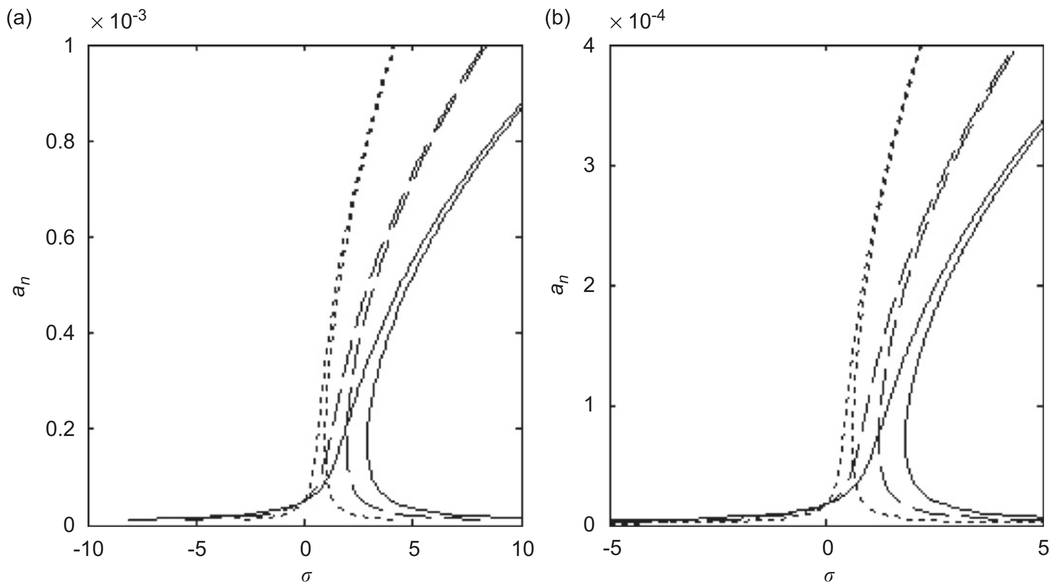

Fig. 7. Comparison of response amplitudes of a different parameter k_2: (a) the first mode and (b) the second mode.

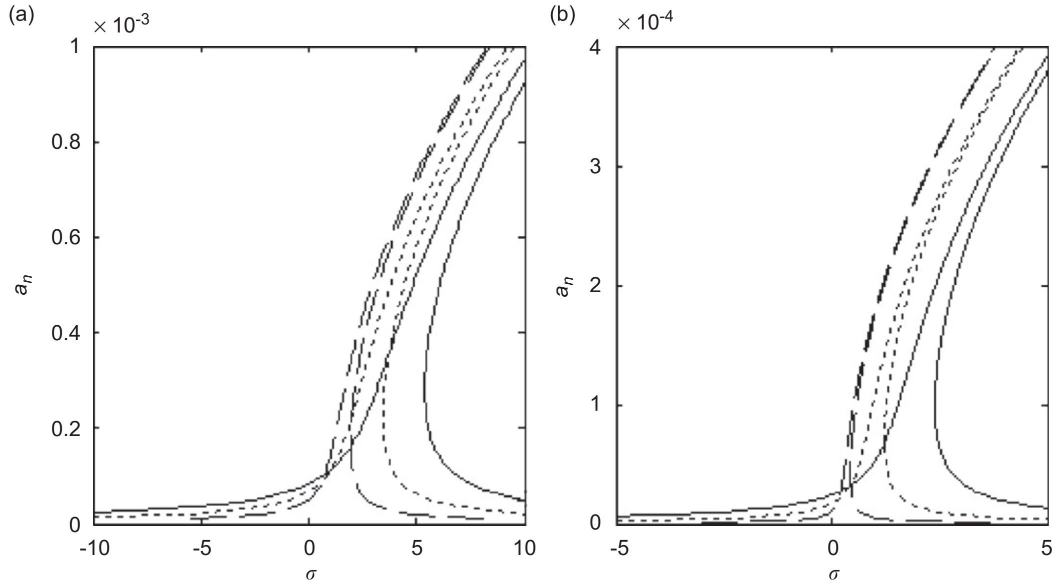

Fig. 8. Comparison of response amplitudes of different excitation amplitudes b: (a) the first mode and (b) the second mode.

Consider an axially moving beam with $k_4 = 0.64$ and $v = 2.0$. The response amplitudes at exact resonance for the first two natural frequencies are shown in Fig. 6. The first two natural frequencies of the unperturbed system are $\omega_1 = 4.7393$ and $\omega_2 = 23.7017$. In the first-mode response, the coefficients are $b = 0.03$, $k_2 = 100{,}000$, and in the second-mode response, the coefficients are $b = 0.5$, $k_2 = 100{,}000$.

Fig. 7 indicates the effects of a different parameter k_2. With an increase of k_2, response under the same conditions decreases. In the first-mode response, the coefficient is $b = 0.03$, and in the second-mode response, the coefficient is $b = 0.5$. The dotted lines are for coefficient $k_2 = 50{,}000$, the dashed for coefficient $k_2 = 100{,}000$, and the solid lines for coefficient $k_2 = 150{,}000$.

The effects of the external excitation amplitudes on the response amplitudes are illustrated in Fig. 8. From the response diagrams, it is clear that the external excitation amplitudes increase the amplitude of the excited system. The coefficient here is $k_2 = 100{,}000$. In the first-mode response, the solid lines are for coefficient $b = 0.05$, the dotted lines for coefficient $b = 0.04$, and the dashed for coefficient $b = 0.03$. In the second-mode response, the solid lines are for coefficient $b = 0.7$, the dotted lines for coefficient $b = 0.5$, and the dashed for coefficient $b = 0.3$.

The stability of the response amplitudes is illustrated in Fig. 9. In the first-mode response, the coefficients are $b = 0.03$ and $k_2 = 100{,}000$. In the second-mode response, the coefficients are $b = 0.7$ and $k_2 = 100{,}000$. The solid lines denote the response amplitudes and the dashed denote the boundary of instability. The inner region of the boundary of instability is unstable and the outer is stable. Thus a jump phenomenon appears.

3.2.3. Subharmonic resonances and responses amplitudes

To study the subharmonic resonances, a detuning parameter σ is introduced to quantify the deviation and ω is described by

$$\omega = 3\omega_0 + \varepsilon\sigma \tag{69}$$

Solvability requires

$$\chi AB\bar{B} + \kappa A^2\bar{A} + \zeta\bar{A}^2 B e^{i\sigma T_1} + \frac{\partial A}{\partial T_1} = 0 \tag{70}$$

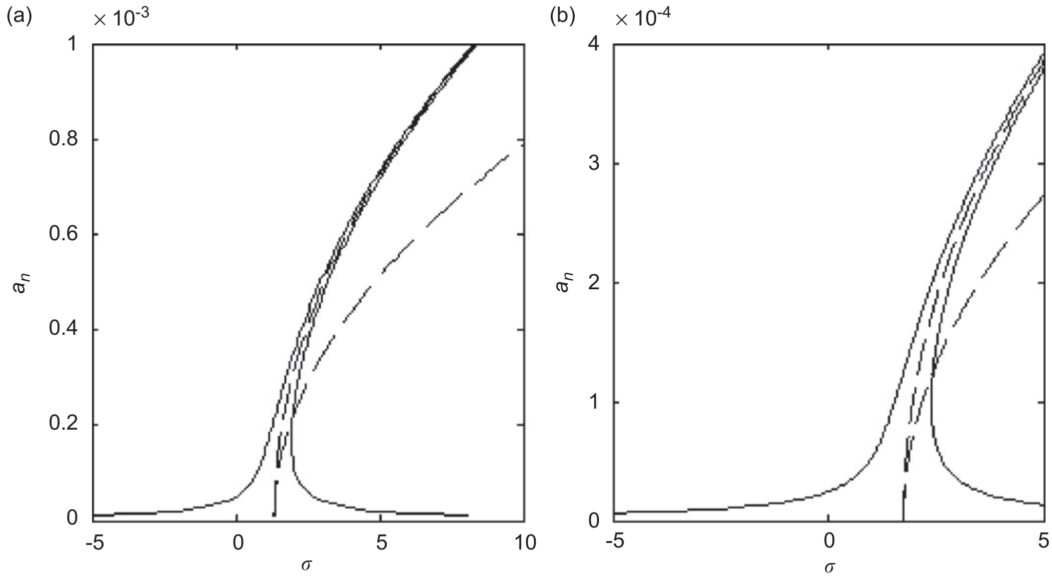

Fig. 9. Stability of the response amplitudes: (a) the first mode and (b) the second mode.

where

$$\chi = 3\Bigg[(k_1\omega_0^2 - k_2)\left(\int_0^1 \bar{\phi}\phi'^2\bar{\phi}''\,\mathrm{d}x + 2\int_0^1 \bar{\phi}\phi'\bar{\phi}'\phi''\,\mathrm{d}x\right) + 6k_1k_2\left(\int_0^1 \bar{\phi}\phi''^2\bar{\phi}''\,\mathrm{d}x + \int_0^1 \bar{\phi}\bar{\phi}'\phi''\phi'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\phi'\bar{\phi}''\phi'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\phi'\phi''\bar{\phi}'''\,\mathrm{d}x\right) + 2k_1k_2 \times \int_0^1 \bar{\phi}\phi'\bar{\phi}'\phi''''\,\mathrm{d}x + k_1k_2\int_0^1 \bar{\phi}\phi'^2\bar{\phi}''''\,\mathrm{d}x\Bigg] \Bigg/ \Bigg\{2\Bigg[\mathrm{i}\omega_0(1-2k_0\omega_0^2) \times \int_0^1 \bar{\phi}\phi\,\mathrm{d}x + v(1-3k_0\omega_0^2)\int_0^1 \bar{\phi}\phi'\,\mathrm{d}x - \mathrm{i}\omega_0(k_0+k_1+k_3-k_0v^2) \times \int_0^1 \bar{\phi}\phi''\,\mathrm{d}x - k_1 v\int_0^1 \bar{\phi}\phi'''\,\mathrm{d}x\Bigg]\Bigg\} \tag{71a}$$

$$\kappa = \frac{1}{2}\chi \tag{71b}$$

$$\zeta = \frac{3}{2}\Bigg[(k_1\omega_0^2 - k_2)\left(\int_0^1 \bar{\phi}\bar{\phi}'^2\phi''\,\mathrm{d}x + 2\int_0^1 \bar{\phi}\phi'\bar{\phi}'\bar{\phi}''\,\mathrm{d}x\right) + 6k_1k_2\int_0^1 \bar{\phi}\phi''\bar{\phi}''^2\,\mathrm{d}x + 6k_1k_2\left(\int_0^1 \bar{\phi}\bar{\phi}'\bar{\phi}''\phi'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\bar{\phi}'\phi''\bar{\phi}'''\,\mathrm{d}x + \int_0^1 \bar{\phi}\phi'\bar{\phi}''\bar{\phi}'''\,\mathrm{d}x\right) + k_1k_2\left(\int_0^1 \bar{\phi}\bar{\phi}'^2\phi''''\,\mathrm{d}x + 2\int_0^1 \bar{\phi}\phi'\bar{\phi}'\bar{\phi}''''\mathrm{d}x\right)\Bigg] \Bigg/ \Bigg\{2\Bigg[\mathrm{i}\omega_0(1-2k_0\omega_0^2)\int_0^1 \bar{\phi}\phi\,\mathrm{d}x + v(1-3k_0\omega_0^2) \times \int_0^1 \bar{\phi}\phi'\,\mathrm{d}x - \mathrm{i}\omega_0(k_0+k_1+k_3-k_0v^2)\int_0^1 \bar{\phi}\phi''\,\mathrm{d}x - k_1 v\int_0^1 \bar{\phi}\phi'''\,\mathrm{d}x\Bigg]\Bigg\} \tag{71c}$$

Consider the transformation

$$A = a_1(T_1, T_2)\,\mathrm{e}^{\mathrm{i}a_2(T_1,T_2)}, \quad B = b_1\,\mathrm{e}^{\mathrm{i}b_2} \tag{72}$$

Substituting Eq. (72) into Eq. (70) yields

$$\frac{\partial a_1}{\partial T_1} = -a_1^3\kappa^R - a_1 b_1^2\chi^R - a_1^2 b_1\left[\zeta^R\cos(b_2+\sigma T_1-3a_2)-\zeta^I\sin(b_2+\sigma T_1-3a_2)\right] \tag{73a}$$

$$\frac{\partial a_2}{\partial T_1} = -a_1^2\kappa^I - b_1^2\chi^I - a_1 b_1\left[\zeta^R\sin(b_2+\sigma T_1-3a_2)+\zeta^I\cos(b_2+\sigma T_1-3a_2)\right] \tag{73b}$$

Then, we change Eq. (73) into an autonomous system

$$\frac{\partial a_1}{\partial T_1} = -a_1^3\kappa^R - a_1 b_1^2\chi^R - a_1^2 b_1(\zeta^R\cos\gamma-\zeta^I\sin\gamma) \tag{74a}$$

$$\frac{\partial \gamma}{\partial T_1} = \sigma + 3a_1^2\kappa^I + 3b_1^2\chi^I + 3a_1 b_1(\zeta^R\sin\gamma+\zeta^I\cos\gamma) \tag{74b}$$

where

$$\gamma = b_2 + \sigma T_1 - 3a_2 \tag{75}$$

For steady-state solutions, the amplitude a_1 and the new phase γ angle in Eq. (74) should be constant. Setting $a_1' = 0$ and $\gamma' = 0$ and then eliminating from Eq. (74) leads to

$$\sigma = -3(a_1^2\kappa^I + b_1^2\chi^I) \pm 3\sqrt{a_1^2 b_1^2(\zeta^{I2}+\zeta^{R2}) - (a_1^2\kappa^R + b_1^2\chi^R)^2} \tag{76}$$

When $a_1 \neq 0$ in Eqs. (74), the **J**-matrix characteristic function of balanced solution on the right hand,

$$\lambda^2 + 2(2a_1^2\kappa^R + b_1^2\chi^R)\lambda + 3(a_1^2\kappa^R + b_1^2\chi^R)(a_1^2\kappa^R - b_1^2\chi^R) - 3\left(\frac{\sigma}{3} + a_1^2\kappa^I + b_1^2\chi^I\right)\left(\frac{\sigma}{3} - a_1^2\kappa^I + b_1^2\chi^I\right) = 0 \tag{77}$$

where λ is an eigenvalue of the system. Based on Routh–Hurwitz theorem, the boundary of instability yields

$$(a_1^2\kappa^R + b_1^2\chi^R)(a_1^2\kappa^R - b_1^2\chi^R) - \left(\frac{\sigma}{3} + a_1^2\kappa^I + b_1^2\chi^I\right)\left(\frac{\sigma}{3} - a_1^2\kappa^I + b_1^2\chi^I\right) = 0 \tag{78}$$

Consider an axially moving beam with $k_4 = 0.64$ and $v = 2.0$. The response amplitudes at frequencies three times those of exact resonance for the first two natural frequencies are shown in Fig. 10. The first two natural

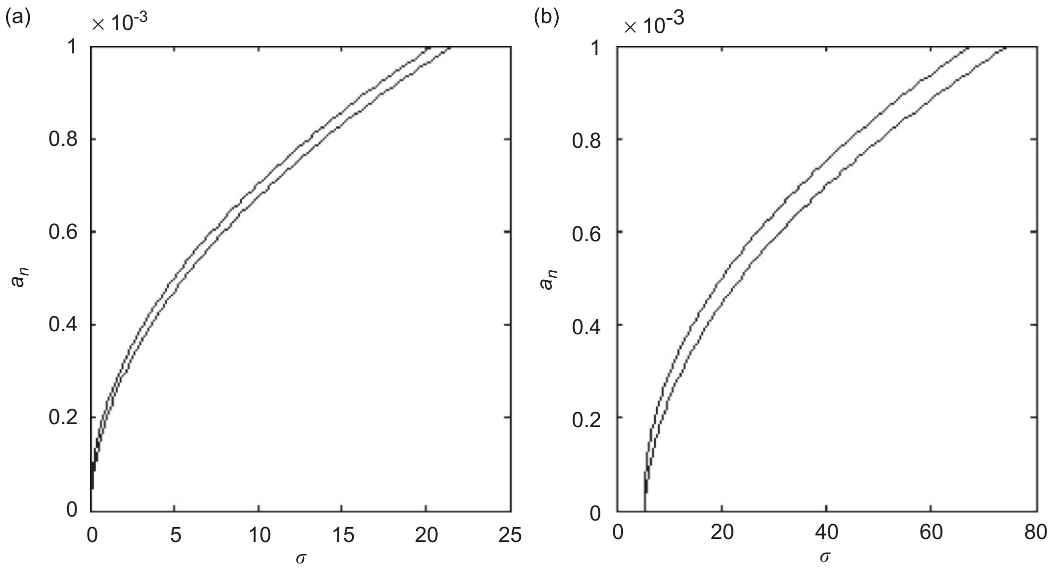

Fig. 10. Response amplitude diagram: (a) the first mode and (b) the second mode.

frequencies of the unperturbed system are $\omega_1 = 4.7393$ and $\omega_2 = 23.7017$. In the first-mode response, the coefficients are $b = 0.03$, $k_2 = 100{,}000$, and in the second-mode response, the coefficients are $b = 5$, $k_2 = 100{,}000$.

Fig. 11 shows the effects of a different parameter k_2. In the first-mode response, the coefficient is $b = 0.03$, and in the second-mode response, the coefficient is $b = 5$. The dotted lines are for coefficient $k_2 = 50{,}000$, the dashed for coefficient $k_2 = 100{,}000$, and the solid lines for coefficient $k_2 = 150{,}000$.

The effects of the external excitation amplitudes on the response amplitudes are illustrated in Fig. 12. From the response diagrams, it is clear that the excitation amplitudes increase the amplitude of the excited system. The coefficient is $k_2 = 100{,}000$. In the first-mode response, the solid lines are for coefficient $b = 0.1$, the dotted

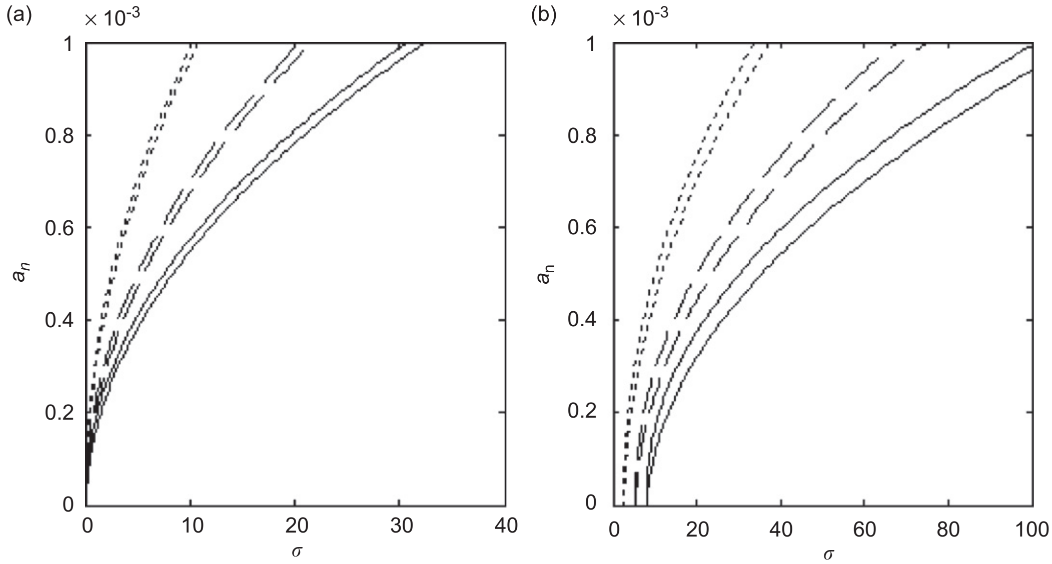

Fig. 11. Comparison of response amplitudes of a different parameter k_2: (a) the first mode and (b) the second mode.

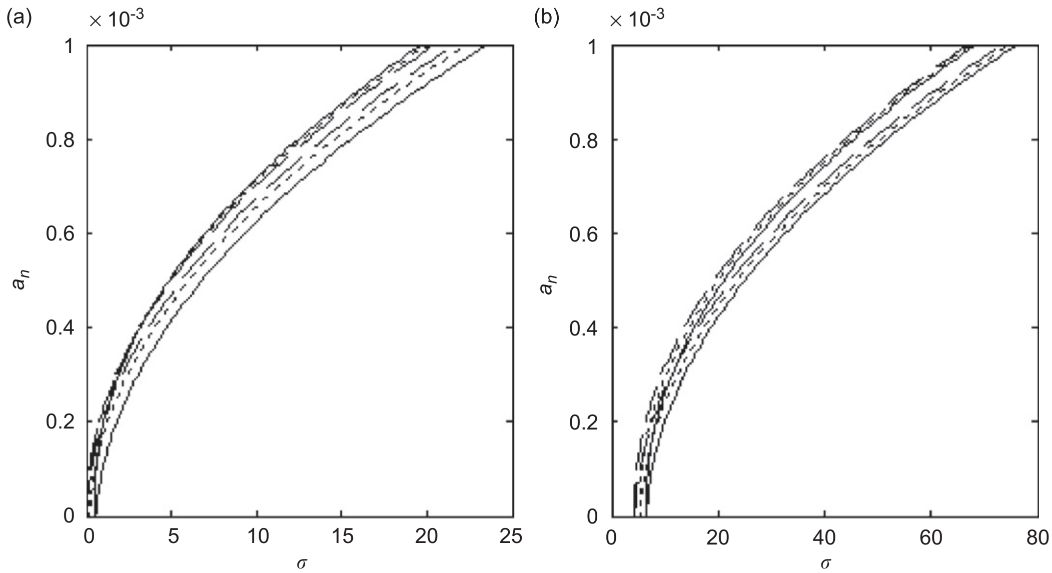

Fig. 12. Comparison of response amplitudes of different excitation amplitudes b: (a) the first mode and (b) the second mode.

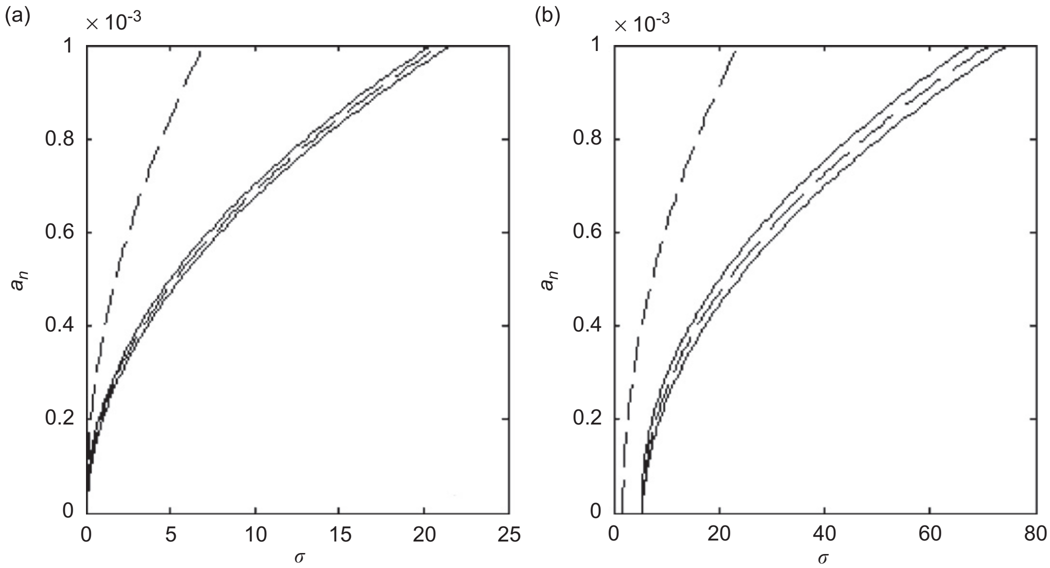

Fig. 13. Stability of the response amplitudes: (a) the first mode and (b) the second mode.

lines for coefficient $b = 0.06$, and the dashed for coefficient $b = 0.03$. In the second-mode response, the solid lines are for coefficient $b = 5.5$, the dotted lines for coefficient $b = 5$, and the dashed for coefficient $b = 4.5$.

The stability of the response amplitudes is illustrated in Fig. 13. In the first-mode response, the coefficients are $b = 0.03$ and $k_2 = 100{,}000$. In the second-mode response, the coefficients are $b = 5$ and $k_2 = 100{,}000$. The solid lines denote the response amplitudes and the dashed denote the boundary of instability. The inner region of the boundary of instability is unstable and the outer is stable. Thus a jump phenomenon appears.

4. Conclusions

In this investigation, nonlinear vibrations under weak and strong external excitations of axially moving beam on simple supports are studied based on the Timoshenko beam model. A partial-differential nonlinear equation is derived from Newton's second law. The multiple-scale method is used to discuss nonsyntonic excitations, superharmonic resonances, and subharmonic resonances. The nontrivial steady-state response and its existence conditions are presented. The system shows a typical muti-valued nonlinear phenomenon. The numerical examples investigated reveal the following.

The natural frequencies decrease with an increase in the axial speed for simple supports. The first natural frequency vanishes at the critical speed and afterwards the system is unstable about the zero equilibrium.

The assumption that the primary response with the possible contributions of modes does not involve resonance is wrong. The modes have only zero stationary solution, which decays to zero exponentially. Therefore, the modes have actually no effect on the stability.

For the efforts of different parameters about nonlinear vibrations under weak and strong external excitations, the response amplitudes increase with decreasing nonlinear effects, and with increasing external excitation amplitudes.

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